ADVANCED GEOMETRY FOR HIGH SCHOOLS

SYNTHETIC—ANALYTICAL

McDOUGALL.

REVISED EDITION

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TORONTO

ADVANCED GEOMETRY

FOR

HIGH SCHOOLS

SYNTHETIC AND ANALYTICAL

PART I.—SYNTHETIC GEOMETRY

PART II.—ANALYTICAL GEOMETRY

BY

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PRINCIPAL OTTAWA COLLEGIATE INSTITUTE

SECOND EDITION

REVISED AND AMPLIFIED

TORONTO

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PREFACE TO THE SECOND EDITION

In this edition changes have been made in Part I. on the suggestion of teachers who have kindly pointed out difficulties experienced by their pupils in using the original edition. The exercises have been re-arranged and, in general, divided into two sections. Those marked (*a*) are intended for a first reading, and are sufficient for candidates for Entrance to the Faculties of Education, while the sections marked (*b*) are more suitable for candidates for honours and scholarships.

Demonstrations have been re-written, diagrams added, and notes given suggesting constructions or proofs. Some typical solutions illustrate methods to be used in maxima and minima.

In many cases, *e.g.*, loci, the difficulties have been diminished. Certain exercises in the First Edition were stated as problems. Under the given conditions the students were asked to find the required solution. These exercises are here changed to theorems. With the same data the conclusion is stated and the proof only is to be discovered.

In both parts considerable additions suitable for honour candidates have been made to the miscellaneous exercises.

Acknowledgments for valuable assistance received from him are due to Mr. I. T. Norris, Mathematical Master of the Ottawa Collegiate Institute, and to the teachers who have given an encouraging approval of the book as well as assistance in making it more useful.

OTTAWA, February, 1919.

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Part I.—SYNTHETIC GEOMETRY

SYNTHETIC GEOMETRY

CHAPTER I

Theorems of Menelaus and Ceva

1. Menelaus' Theorem:—If a transversal cut the sides, or the sides produced, of a triangle, the product of one set of alternate segments taken in circular order is equal to the product of the other set.

(Note.—*The transversal must cut two sides and the third side produced, or cut all three produced.*)

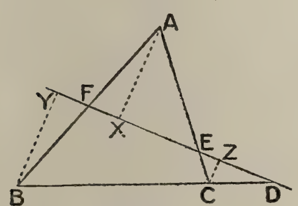

Fig. 1.

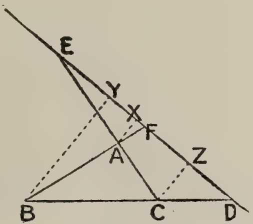

Fig. 2.

A transversal cuts the sides **BC, CA, AB** of the △ **ABC** in the points **D, E, F** respectively, then

$$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

Draw **AX, BY, CZ** ⊥ to the transversal.

From similar △s:—

$$\frac{AF}{FB} = \frac{AX}{BY},$$

$$\frac{BD}{DC} = \frac{BY}{CZ},$$

and $$\frac{CE}{EA} = \frac{CZ}{AX}.$$

By multiplication,

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1,$$

and $\therefore$ $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$

(Historical Note.—Menelaus, a Greek, lived in Alexandria, Egypt, about 98 A.D.)

2. **Converse of Menelaus' Theorem:**—If, in $\triangle$ ABC on two of the sides BC, CA, AB and on the third produced, or if on all three produced, points D, E, F respectively be taken so that AF . BD . CE = FB . DC . EA, the points D, E, F are collinear.

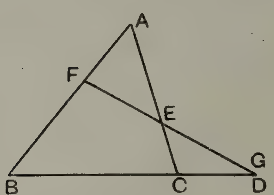

FIG. 3.

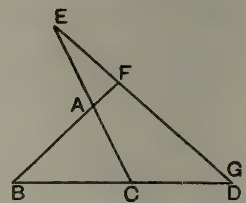

FIG. 4.

Join EF, and produce EF to cut BC at G.

$\because$ FEG is a st. line,

$\therefore$ AF . BG . CE = FB . GC . EA. (§ 1.)

But AF . BD . CE = FB . DC . EA, by hypothesis.

$$\therefore \text{, dividing, } \frac{BG}{BD} = \frac{GC}{DC};$$

$$\text{or, by alternation, } \frac{BG}{GC} = \frac{BD}{DC}.$$

$\therefore$ G coincides with D. (O.H.S. Geometry, § 121.)

$\therefore$ D, E, F are collinear.

3. **Ceva's Theorem:—If from the vertices of a triangle concurrent straight lines be drawn to cut the opposite sides, the product of one set of alternate segments taken in circular order is equal to the product of the other set.**

(NOTE.—D, E *and* F *must be on the three sides, or on one side and on the other two produced.*)

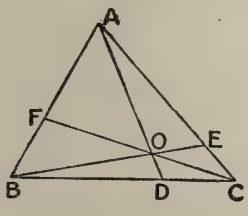

FIG. 5.

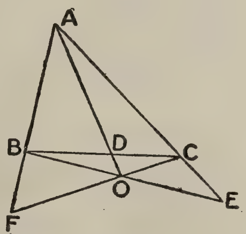

FIG. 6.

AO, BO, CO drawn from the vertices of △ ABC cut BC, CA, AB, at D, E, F respectively, then

$$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

FOC is a transversal of △ ABD,

$\therefore$ AF . BC . DO = FB . CD . OA. (§ 1.)

BOE is a transversal of △ ADC,

$\therefore$ AO . DB. CE = OD . BC . EA.

By multiplication, and division by DO, OA and BC,

$$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

(For another proof of this theorem see O. H. S. Geometry, § 122, Exercises 12 and 14.)

(HISTORICAL NOTE.—Giovanni Ceva, an Italian engineer, died in 1734 A.D.)

4. **Converse of Ceva's Theorem:—If, in △ ABC, on the three sides BC, CA, AB, or if on one of these sides and on the other two produced, points D, E, F respectively be taken so that AF . BD . CE = FB . DC . EA, the lines AD, BE, CF are concurrent.**

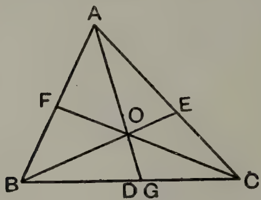

Fig. 7.

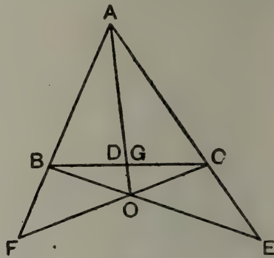

Fig. 8.

Draw BE, CF and let them cut at O. Join AO and let it cut BC at G.

∵ AG, BE, CF are concurrent,

∴ AF . BG . CE = FB . GC . EA, (§ 3.)

But AF . BD . CE = FB . DC . EA, by hypothesis.

∴, dividing, $\frac{BG}{BD} = \frac{GC}{DC}$;

or, by alternation, $\frac{BG}{GC} = \frac{BD}{DC}$.

∴ G coincides with D.

∴ AD, BE, CF are concurrent.

5. **The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.**

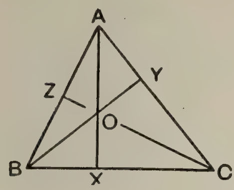

FIG. 9.

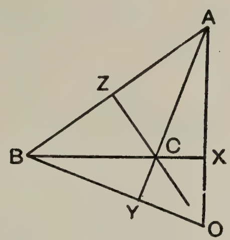

FIG. 10.

In $\triangle$ ABC, draw AX $\perp$ BC, BY $\perp$ CA, CZ $\perp$ AB.

To prove that AX, BY, CZ are concurrent.

$$\because \qquad \triangle AZC \mathrel{|||} \triangle AYB,$$

$$\therefore \qquad \frac{AZ}{YA} = \frac{CA}{AB}.$$

Similarly, $$\frac{BX}{ZB} = \frac{AB}{BC};$$

and $$\frac{CY}{XC} = \frac{BC}{CA}.$$

$$\therefore \quad \frac{AZ}{YA} \cdot \frac{BX}{ZB} \cdot \frac{CY}{XC} = \frac{CA}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{CA} = 1.$$

$$\therefore \qquad AZ \,.\, BX \,.\, CY = ZB \,.\, XC \,.\, YA.$$

$\therefore$, by § 4, AX, BY, CZ are concurrent.

6. The point where the $\perp$s from the vertices of a $\triangle$ to the opposite sides intersect is called the **ortho-centre** of the $\triangle$. The $\triangle$ formed by joining the feet of these $\perp$s, X, Y, Z in Figures 9 or 10, is called the **orthocentric,** or **pedal,** $\triangle$.

7.—Exercises

(*a*)

1. Show, from the converse of Ceva's Theorem, that the medians of a △ are concurrent.

2. In △ **ABC**, the bisectors of the ∠s **A, B, C** cut **BC, CA, AB** at **D, E, F** respectively. Show that $\mathbf{AF} = \dfrac{bc}{a+b}$.

If $a = 25$, $b = 35$, $c = 20$, show that $\mathbf{AF} \cdot \mathbf{BD} \cdot \mathbf{CE} = 2062\frac{86}{297}$.

3 Show, from the converse of Ceva's Theorem, that the bisectors of the ∠s of a △ are concurrent.

4. In △ **ABC**, the bisector of the interior ∠ at **A** and of the exterior ∠s at **B, C** cut **BC, CA, AB** at **D, E, F** respectively. Show that $\mathbf{AF} = \dfrac{bc}{b-a}$.

If $a = 44$, $b = 33$, $c = 22$, show that $\mathbf{AF} \cdot \mathbf{BD} \cdot \mathbf{CE} = 76665\frac{3}{5}$.

5 Show, from the converse of Ceva's Theorem, that the bisector of the ∠ at one vertex of a △ and the bisectors of the exterior ∠s at the other two vertices are concurrent.

6. In △ **ABC, AX, BY, CZ** the ⊥s to **BC, CA, AB** intersect at **O**. Show that:—

(*a*) rect. **AO . OX** = rect. **BO . OY** = rect. **CO . OZ**;

(*b*) rect. **AB . AZ** = rect. **AO . AX** = rect. **AC . AY**;

(*c*) if **AX** meet the circumscribed circle of △ **ABC** at **K**, **OX** = **XK**;

(*d*) ∠**AYZ** = ∠**B** = ∠**AOZ**;

(*e*) △s **AYZ, BZX, CXY, ABC** are similar;

(*f*) **AX, BY, CZ** bisect the ∠s of the pedal △ **XYZ**;

(*g*) of the four points **A, B, C, O**, each is the orthocentre of the △ of which the other three points are the vertices;

(*h*) if a △ **LMN** be formed by drawing through **A, B, C** lines **MN, NL, LM** ∥ **BC, CA, AB** respectively, **O** is the circumscribed centre of △ **LMN**;

(*i*) if **S** be the centre of the circumscribed circle of △ **ABC, AS, BS, CS** are respectively ⊥ **YZ, ZX, XY** the sides of the orthocentric △.

7. In △ **ABC**, the inscribed circle touches **BC, CA, AB** at **D, E, F** respectively, and s = the semi-perimeter. Show that **AF** $= s - a$.

If $a = 43$, $b = 31$, $c = 26$, show that **AF . BD . CE** $= 3192$.

8. The st. lines joining the vertices of a △ to the points of contact of the opposite sides with the inscribed circle are concurrent.

9. In △ **ABC**, an escribed circle touches **BC** at **D** and **AB, AC**, produced at **F, E** respectively. Show that **FB** $= s - c$.

If $a=40$, $b=30$, $c=50$, show that **AF . BD . CE** $=18000$.

10. The st. lines joining the vertices of a △ to the points of contact of the opposite sides with any one of the escribed circles are concurrent.

11. **O** is a point within the △ **ABC** and **AO, BO, CO** produced cut **BC, CA, AB** at **D, E, F** respectively. The circle through **D, E, F** cuts **BC, CA, AB** again at **P, Q, R.** Show that **AP, BQ, CR** are concurrent.

(*b*)

The bisectors of ∠s **B, C** of △ **ABC** cut **CA, AB** at **F** respectively. **FE, BC** produced meet at **D.** Prove that **AD** bisects the exterior ∠ at **A.**

13. The points where the bisectors of the exterior ∠s at **A, B, C** of △ **ABC** meet **BC, CA, AB** respectively are collinear.

14. **AB, CD, EF** are three || st. lines. **AC, BD** meet at **N**; **CE, DF** at **L**; **EA, FB** at **M**. Prove that **L, M, N** are collinear.

15. (*a*) If two △s are so situated that the st. lines joining their vertices in pairs are concurrent, the intersections of pairs of corresponding sides are collinear.—*Desargues' Theorem.*

(Note.—**ABC**, *abc* the two △s; **A***a*, **B***b*, **C***c* meeting at **O**; **BC**, *bc* at **L**; **CA**, *ca* at **M**; **AB**, *ab* at **N**. Using the △s **OBC, OCA, OAB** and the respective transversals *bc***L**, *ac***M**, *ab***N** prove that **AN . BL . CM = NB . LC . MA**.)

(*b*) State and prove the converse of (*a*).

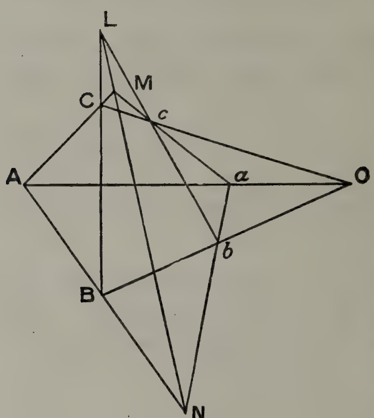

Fig. 11.

(Note.—**BC**, *bc* meet at **L**; **CA**, *ca* at **M**; **AB**, *ab* at **N** and **L, M, N** are collinear. Produce **A***a*, **B***b* to meet at **O**. To show that **C***c* passes through **O**. **A***a***M**, **B***b***L** are △s having **AB**, *ab*, **ML** concurrent at **N**; corresponding sides *a***M**, *b***L** meet at *c*; **MA, LB** at **C**; **A***a*, **B***b* at **O**. ∴, by (*a*), *c*, **C, O** are collinear, *i.e.*, **A***a*, **B***b*, **C***c* are concurrent.)

(Historical Note:—Girard Desargues (1593-1662) was an architect and engineer of Lyons, France.)

16. The inscribed circle of △ **ABC** touches the sides **BC, CA, AB** at **D, E, F** respectively; **EF, FD, DE**, produced meet **BC, CA, AB** respectively at **L, M, N**. Show that **L, M, N** are collinear.

(Note.—Use Ex. 8 and Ex. 15 (*a*).)

17. Tangents to the circumcircle at **A, B, C,** meet **BC, CA, AB** respectively in collinear points.

(NOTE.—Use Ex. 8 and Ex. 15 (*a*).)

18. Given the base and vertical ∠ of a △, find the locus of its orthocentre.

19. If the base **BC** and vertical ∠ **A** of a △ **ABC** be given, and the base be trisected at **D, E,** the locus of the centroid is an arc containing an ∠ equal to ∠ **A,** and having **DE** as its chord.

20. **ABC** is a △, **XYZ** its pedal △. Show that the respective intersections of **BC, CA, AB** with **YZ, ZX, XY** are collinear.

21. Where is the orthocentre of a rt.-∠d △?

22. If one escribed circle of △ **ABC** touch **AC** at **F** and **BA** produced at **G,** and another escribed circle touch **AB** at **H** and **CA** produced at **K, FH, KG** produced cut **BC** produced in points equidistant from the middle point of **BC.**

(NOTE.—Use § 1, taking the two transversals **FH, KG** of △ **ABC** and multiplying the results.)

23. If **O** is the orthocentre, **S** the circumcentre and **I** the centre of the inscribed circle of △ **ABC,** prove that **IA** bisects ∠ **OAS.**

8. The distance from each vertex of a triangle to the orthocentre is twice the perpendicular from the circumcentre to the side opposite that vertex.

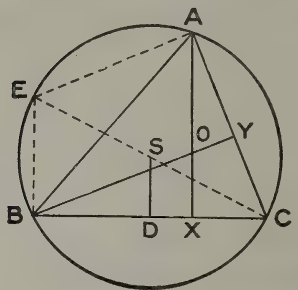

FIG. 12.

O is the orthocentre, S the circumcentre of △ ABC; SD is ⊥ BC.

To show that AO = twice SD.

Draw the diameter CSE, join BE, EA.

∠s EBC, EAC being ∠s in semicircles are rt. ∠s.

∴ EB ∥ AO and EA ∥ BY.

∴ EAOB is a ∥gm, and EB = AO.

But ∵ S, D are middle points of EC, BC,

∴ EB = twice SD,

and ∴ AO = twice SD.

9. ABC is a △ having AX ⊥ BC, AD a median, O the orthocentre, and S the circumscribed centre. Show that OS cuts AD at the centroid G. (Use § 8.) Show also that G is a point of trisection in SO.

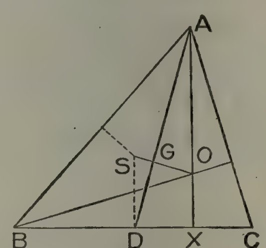

FIG. 13.

A general enunciation of these results may be given as follows:—

The Orthocentre, Centroid, and the centre of the Circumscribed Circle of a △ are in the same st. line, and the Centroid is a point of trisection in the st. line joining the other two.

The Nine-Point Circle

10. The three middle points of the sides of a triangle, the three projections of the vertices on the opposite sides, and the three middle points of the straight lines joining the vertices to the orthocentre are all concyclic.

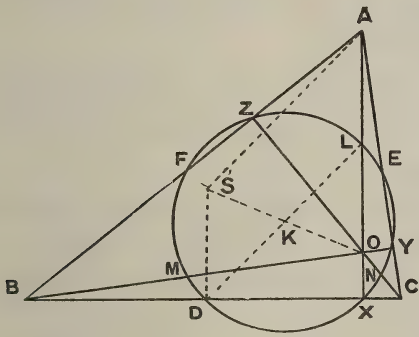

Fig. 14.

Let ABC be a △, AX, BY, CZ the ⊥s from A, B, C to BC, CA, AB respectively, O the orthocentre, L, M, N the middle points of AO, BO, CO respectively, D, E, F the middle points of BC, CA, AB respectively.

It is required to show that the nine points, X, Y, Z, L, M, N, D, E, F are concyclic.

Find S, the circumcentre. Draw SO, SD, SA. Bisect SO at K. Draw KL.

∵ K, L are the middle points of SO, OA;

∴ KL = half of SA;

and ∴ the circle described with centre K and radius equal to half that of the circumcircle passes through L.

Similarly this circle passes through M and N.

Draw DK.

In $\triangle$s SKD, OKL $\left\{\begin{array}{ll} \text{SD} = \text{LO}, & (\S\,8) \\ \text{SK} = \text{KO}, & \\ \angle\text{DSK} = \angle\text{KOL}, & (\text{SD} \parallel \text{LO}); \end{array}\right.$

$\therefore$ DK = KL and $\angle$SKD = $\angle$OKL.

$\because$ DK = KL; the circle with centre K and radius KL passes through D.

Similarly this circle passes through E and F.

$\because$ $\angle$SKD = $\angle$OKL, and SKO is a st. line;

$\therefore$ LKD is a st. line

$\therefore$ K is the middle point of the hypotenuse of the rt.-$\angle$d $\triangle$ LDK;

$\therefore$ the circle with centre K and radius KL passes through X.

Similarly this circle passes through Y and Z.

$\therefore$ the nine points L, M, N, D, E, F, X, Y, Z are concyclic.

Cor. 1:—The centre of the N.-P. circle is the middle point of the line joining the circumcentre to the orthocentre.

Cor. 2:—The diameter of the N.-P. circle is equal to the radius of the circumcircle.

SIMPSON'S LINE

11. If any point is taken on the circumference of the circumscribed circle of a triangle, the projections of this point on the three sides of the triangle are collinear.

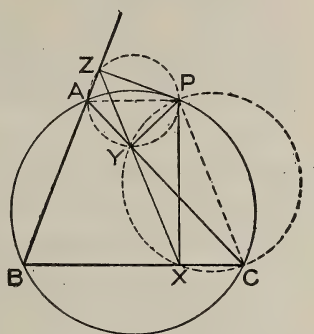

FIG. 15.

Let P be any point on the circle ABC, X, Y, Z, the projections of P on BC, CA, AB respectively.

It is required to show that X, Y, Z are in the same st. line.

Join ZY, YX, PC, PA.

$\angle$ PYC = $\angle$ PXC, $\therefore$ P, Y, X, C, are concyclic, and $\angle$ XYC = $\angle$ XPC.

$\angle$ AZP + $\angle$ AYP = 2 rt. $\angle$s, $\therefore$ A, Z, P, Y are concyclic, and $\angle$ AYZ = $\angle$ APZ.

APCB is a cyclic quadrilateral,

$\therefore$ $\angle$ APC + $\angle$ B = 2 rt. $\angle$s.

In quadrilateral BZPX,

$\angle$s BZP, BXP are rt. $\angle$s,

$\therefore$ $\angle$ ZPX + $\angle$ B = 2 rt. $\angle$s.

Hence $\angle$ ZPX = $\angle$ APC, and as the part APX is common to these $\angle$s,

$\angle$ APZ = $\angle$ XPC.

$\therefore$ $\angle$ AYZ = $\angle$ XYC, and

$\angle$ XYC + $\angle$ CYZ = $\angle$ AYZ + CYZ = 2 rt. $\triangle$s;

$\therefore$ XY and YZ are in the same st. line.

(HISTORICAL NOTE.—Robert Simpson (1687-1768) was Professor of Mathematics in the University of Glasgow.)

12.—Exercises

(*a*)

1. **P** is the orthocentre of △ **DEF**, and the ||gm **EPFG** is completed. Show that **DG** is a diameter of the circle circumscribing **DEF.**

2. **ABC** is a △; **L, M, N** the centres of its escribed circles. Show that the circle circumscribed about **ABC** is the N.-P. circle of △ **LMN.**

3. In △ **ABC, I** is the centre of the inscribed circle, **L, M, N** the centres of the escribed circles. Prove that the circumcircle of △ **ABC** bisects **IL** and **LM.**

4. **O** is the orthocentre of △ **ABC.** Prove that △s **OBC, ABC** have the same N.-P. circle.

5. Given the base and vertical ∠ of a △, show that the locus of the centre of its N.-P. circle is a circle having its centre at the middle point of the base.

(Note.—Using Cor. 2 of § 10 show that the distance of **K** from the fixed point **D** is constant.)

6. If the projections of a point on the sides of a △ are collinear, the point is on the circumcircle of the △.

7. The three circles which go through two vertices of a △ and its orthocentre are each equal to the circle circumscribed about the △.

8. The ⊥ from the middle point of a side of a △ on the opposite side of the pedal △ bisects that side.

9. Construct a △ given a vertex, the circumcircle and the orthocentre.

(Note.—Describe the circumcircle and produce **AO** to cut it in **K,** where **A** is the given vertex and **O** is the orthocentre. Draw the rt. bisector of **OK** meeting the circle in **B, C.** See § 7, Ex. 6 (*c*).)

10. **DEF** is a $\triangle$ and **O** is its orthocentre. About **DOF** a circle is described and **EO** is produced to meet the cirumference at **P.** Show that **DF** bisects **EP.**

11. **I** is the centre of the inscribed circle of $\triangle$ **ABC**, and **AI, BI, CI** are produced to meet the circumcircle at **L, M, N.** Prove that **I** is the orthocentre of $\triangle$ **LMN.**

12. **I** is the centre of the inscribed circle of $\triangle$ **ABC**, and the circumcircle of $\triangle$ **IBC** cuts **AB** at **D.** Prove that **AD = AC.**

13. In $\triangle$ **ABC**, the $\perp$s from **A, B** to the opposites sides meet the circumcircle at **D, E.** Show that arc **CD** = arc **CE.**

14. **XYZ** is the pedal $\triangle$ of $\triangle$ **ABC.** Prove that **A, B, C** are the centres of the escribed circles of $\triangle$ **XYZ.**

(*b*)

15. **X, Y, Z** are the projections of **A, B, C** on **BC, CA, AB.** Prove that

(*a*) $YZ \, . \, ZX = AZ \, . \, ZB$;

(*b*) $YZ \, . \, ZX \, . \, XY = AZ \, . \, BX \, . \, CY$.

16. Given the base and vertical $\angle$ of a $\triangle$, to find the loci of the centres of the escribed circles. Let **BC** be the base and **BDC** a segment of a circle containing $\angle$ **BDC** = the given vertical $\angle$. Draw the rt. bisector of **BC** cutting the circle **BDC** at **D, E.** Prove the loci are arcs on the chord **BC** and having their centres at **D, E.**

17. Construct a $\triangle$ having given the base, the vertical $\angle$ and the radius of an escribed circle. (Two cases.)

(Note.—Construct the loci as in Ex. 16. In one case, on the arc with centre **E**, find the point I_1 such that its distance from **BC** equals the given radius. Draw I_1E and produce to cut the arc **BDC** at **A.** **ABC** is the required $\triangle$.

In the other case on the arc with centre **D** find the point I_2 such that its distance from **BC** equals the given radius. Draw I_2D cutting the arc **BDC** at **A′**. **A′BC** is the required △.)

18. **O** is the orthocentre of △ **ABC**, and **D**, **E**, **F** are the circumcentres of △s **BOC**, **COA**, **AOB**. Show that

(*a*) △ **DEF** ≡ △ **ABC**;

(*b*) The orthocentre of each of the △s **ABC**, **DEF** is the circumcentre of the other;

(*c*) The two △s have the same N.-P. circle.

(NOTE.—The rt. bisectors of **AO**, **BO**, **CO** form the sides of △ **DEF**. Let **L** be the middle point of **AO**. ⊥s from **D**, **E** on **BC**, **CA** respectively meet at **S** the circumcentre of △ **ABC**. **DS** bisects **BC** at **G**. Prove ∠ **LEO** = ∠ **OCA** = ∠ **GCS** and comparing △s **OLE**, **GSC**, using § 8, show **LE** = **GC**. Similarly **LF** = **BG**, so that **FE** = **BC**, etc.)

19. Find a point such that its projections on the four sides of a given quadrilateral are collinear.

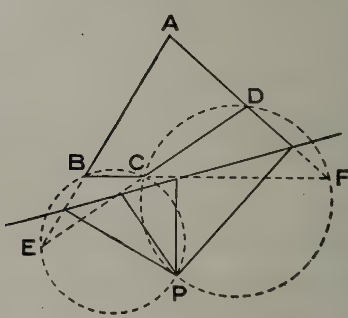

FIG. 16.

(NOTE.—**ABCD** the given quadrilateral; produce the opposite sides to meet at **E**, **F**. Describe circles about △s **EBC**, **FCD** meeting again at **P**. **P** is the required point.

20. In the $\triangle$ **ABC**, the $\perp$ from **A** to **BC** is produced to cut the circumcircle at **P**. Prove that the Simpson's Line of **P** is $\parallel$ to the tangent to the circumcircle at **A**.

21. **P** is any point on the circumcircle of $\triangle$ **ABC**. The $\perp$s from **P** to the sides of the $\triangle$ meet the circle at **D, E, F**. Prove that $\triangle$ **DEF** $\equiv \triangle$ **ABC**.

22. **P** is any point on the circumcircle of a $\triangle$ **ABC** of which **O** is the orthocentre and **X** the projection of **A** on **BC**; **AX** produced cuts the circumcircle at **D** and **PD** cuts **BC** at **E**. Prove that the Simpson's Line of **P** bisects **PE**, is $\parallel$ **OE**, and bisects **OP**.

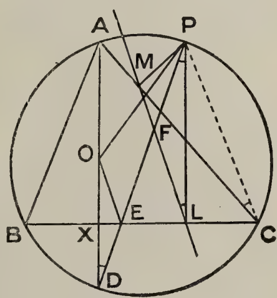

FIG. 17.

(NOTE.—**ML** cuts **PE** at **F**. Draw **PC**.

$\angle$ **FPL** = $\angle$ **FDA** = $\angle$ **PCA** = $\angle$ **PLF**;

$\therefore$ **FL** = **FP**. Then **F** is the middle point of the hypotenuse of the rt.-$\angle$d $\triangle$ **PLE**.

$\therefore$ Simpson's Line bisects **PE**.

$\because$ $\angle$**FLP** = $\angle$**XDE**; $\therefore$ $\angle$**FLE** = $\angle$ **XED**. But $\angle$ **OEX** = $\angle$ **XED** (See § 7, Ex. 6 (*c*).

$\therefore$ $\angle$ **OEX** = $\angle$ **FLE**, and **LM** $\parallel$ **OE**.

In $\triangle$ **POE**, **LM** $\parallel$ **OE** and bisects **PE**; $\therefore$ **LM** bisects **PO**.)

Give a general statement of the last of the three results in Ex. 22.

Areas of Rectangles

13. **If from the vertex of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circumcircle of the triangle.**

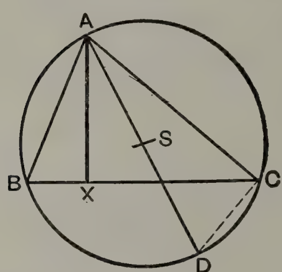

Fig. 18.

$AX \perp BC$ and AD is a diameter of the circumcircle of $\triangle ABC$.

To prove that

rect. $AB \cdot AC$ = rect. $AX \cdot AD$.

Join DC.

$\because \angle AXB = \angle ACD$,

and $\angle ABX = \angle ADC$.

$\therefore \triangle AXB \parallel\!| \triangle ACD$.

$$\therefore \quad \frac{AB}{AD} = \frac{AX}{AC}.$$

$\therefore$ rect. $AB \cdot AC$ = rect. $AX \cdot AD$.

14. **If the vertical angle of a triangle is bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the straight line which bisects the angle.**

ABC is a $\triangle$ and AD the bisector of $\angle A$.

It is required to show that the rect. $AB \cdot AC$ = rect. $BD \cdot DC + AD^2$.

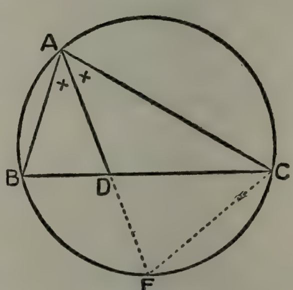

Fig. 19.

Circumscribe a circle about the $\triangle ABC$. Produce AD to cut the circumference at E. Join EC.

In $\triangle$s BAD, EAC $\angle$ BAD = $\angle$ EAC, $\angle$ ABD = $\angle$ AEC,

$\therefore$ $\angle$ ADB = $\angle$ ACE and the $\triangle$s are similar;

$$\text{hence} \quad \frac{BA}{AD} = \frac{EA}{AC},$$

$$\text{and} \quad \therefore \ BA \,.\, AC = AD \,.\, EA.$$

But
$$\begin{aligned} AD \,.\, EA &= AD\,(AD + DE) \\ &= AD^2 + AD \,.\, DE \\ &= AD^2 + BD \,.\, DC. \end{aligned}$$

$\therefore$ rect. BA . AC = rect. BD . DC + AD^2.

15. **Ptolemy's Theorem:—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.**

This theorem is a particular case of that of § *16.*

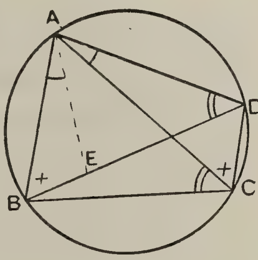

FIG. 20.

ABCD is a quadrilateral inscribed in a circle.

To prove that

AC . BD = AB . CD + BC . AD.

Make $\angle$ BAE = $\angle$ CAD, and produce AE to cut BD at E.

$$\because \quad \triangle ABE \,|||\, \triangle ACD,$$

$$\therefore \quad \frac{AB}{AC} = \frac{BE}{CD}.$$

$$\therefore \quad AB \,.\, CD = AC \,.\, BE.$$

$$\because \quad \triangle ADE \,|||\, \triangle ABC,$$

$$\therefore \quad \frac{AD}{AC} = \frac{DE}{BC}.$$

$$\therefore \quad AD \,.\, BC = AC \,.\, DE.$$

$$\begin{aligned} \therefore \ AB \,.\, CD + AD \,.\, BC &= AC \,.\, BE + AC \,.\, DE \\ &= AC\,(BE + ED) \\ &= AC \,.\, BD. \end{aligned}$$

(HISTORICAL NOTE.—Ptolemy, a native of Egypt, flourished in Alexandria in 139 A.D.)

16. The sum of the rectangles contained by the opposite sides of a quadrilateral is not less than the rectangle contained by the diagonals.

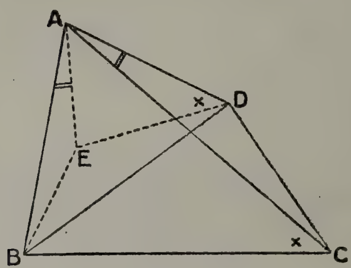

Fig. 21.

ABCD is a quadrilateral, AC, BD its diagonals.

Required to show that AB . DC + AD . BC is not less than AC . BD.

Make $\angle$ BAE = $\angle$ CAD and $\angle$ ADE = $\angle$ ACB. Join EB.

$\triangle$s BAC, EAD are similar;

$$\therefore \quad \frac{BA}{AE} = \frac{CA}{AD}$$

and $\angle$ BAE = $\angle$ CAD,

$\therefore$ $\triangle$s BAE, CAD are also similar.

From the similar $\triangle$s BAE, CAD

$\frac{AB}{BE} = \frac{AC}{CD}$ and $\therefore$ AB . CD = AC . BE.

From the similar $\triangle$s BAC, EAD,

$\frac{BC}{AC} = \frac{ED}{AD}$, and $\therefore$ BC . AD = AC . ED.

Consequently AB . CD + BC . AD = AC (BE + ED);

but BE + ED is not < BD;

$\therefore$ AB . CD + BC . AD is not < AC . BD.

17.—Exercises

(*a*)

1. If the exterior vertical $\angle$ **A** of $\triangle$ **ABC** be bisected by a line which cuts **BC** produced at **D**, rect. **AB . AC** = rect. **BD . CD** $-$ **AD**^{2}.

(NOTE.—Draw the circle **ACB.** Produce **DA** to cut the circumference at **E.** Draw **EC.** Prove $\triangle$ **BAD** $|||$ $\triangle$ **EAC,** and that $\therefore \frac{BA}{AD} = \frac{EA}{AC}$. Then **BA . AC** = **AD . EA** = **AD (ED** $-$ **AD)** = **AD . ED** $-$ **AD**^{2} = **BD . CD** $-$ **AD**^{2}.)

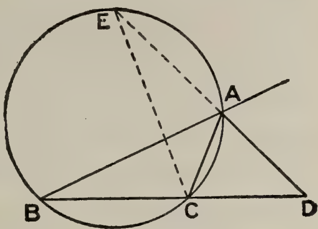

FIG. 22.

2. Draw $\triangle$ **ABC** having a = 81 mm., b = 60 mm., c = 30 mm. Bisect the interior and exterior $\angle$s at **A** and produce the bisectors to meet **BC** and **BC** produced at **D** and **E.** Measure **AD, AE**; and check your results by calculation.

3. If the internal and external bisectors of $\angle$ **A** of $\triangle$ **ABC** meet **BC** at **L, M** respectively, prove

$$LM^2 = BM . MC - BL . LC.$$

4. If **R** is the radius of the circumcircle and $\triangle$ the area of $\triangle$ **ABC,** prove that

$$R = \frac{abc}{4 \triangle}.$$

If the sides of a $\triangle$ are 39, 42, 45, show that **R** = $24\frac{3}{8}$.

5. **P** is any point on the circumcircle of an equilateral $\triangle$ **ABC.** Show that, of the three distances **PA, PB, PC,** one is the sum of the other two.

(*b*)

6. From any point **P** on a circle $\perp$s are drawn to the four sides and to the diagonals of an inscribed quadrilateral. Prove that the rect. contained by the $\perp$s on either pair of

opposite sides is equal to the rect. contained by the $\perp$s on the diagonals.

7. With given base and vertical $\angle$ construct a $\triangle$ having the rect. contained by its sides equal to the square on a given st. line.

8. **A, B, C, D** are given points on a circle. Find a point **P** on the circle such that **PA . PC = PB . PD.**

9. **AB** is the chord of contact of tangents drawn from a point **P** to a circle. **PCD** cuts the circle at **C, D.** Prove that **AB . CD** = 2 **AC . BD.**

10. **I** is the centre of the inscribed circle of $\triangle$ **ABC. AI** produced meets the circumcircle at **K.** Prove **AI . IK** = 2**R**r.

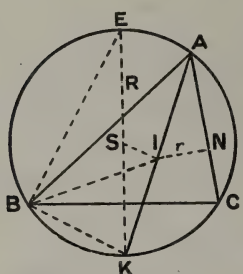

Fig. 23.

(Note.—Draw the diameter **KE** of circle **ABC** and the radius **IN** of the inscribed circle. Draw **BE, BK, BI.**

Show that $\triangle$ **BEK** ||| $\triangle$ **NAI,** and that $\therefore \frac{AI}{IN} = \frac{KE}{KB}$;

or, **AI . KB** = 2**R**r.

Show that **KB = KI,** and that $\therefore$ **AI . IK** = 2**R**r.)

Hence, using Ex. 6, Page 256, O.H.S. Geometry, show that, if **S** be the circumcentre of $\triangle$ **ABC,** $SI^2 = R^2 - 2Rr$.

11. I_1 is the centre of the escribed circle opposite to **A** in $\triangle$ **ABC.** AI_1 cuts the circumcircle **ABC** at **K.** Prove $AI_1 . I_1K = 2Rr_1$.

Hence show that, if **S** be the circumcentre of $\triangle$ **ABC,** $SI_1^2 = R^2 + 2Rr_1$.

Radical Axis

18. The locus of the points from which tangents drawn to two circles are equal to each other is called the **radical axis** of the two circles.

19. **If two circles cut each other, their common chord produced is the radical axis.**

[*Proof left for the pupil.*]

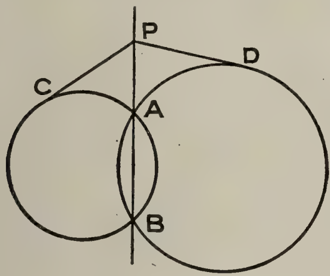

Fig. 24.

20. The locus of a point P such that the difference of the squares of its distances from two fixed points A, B is constant is a st. line perpendicular to AB.

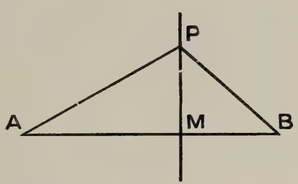

Fig. 25.

From P draw PM $\perp$ AB. Let AB $= a$, AM $= x$ and $PA^2 - PB^2 = k$, where a and k are constants.

$$AM^2 + MP^2 = PA^2$$
$$MB^2 + MP^2 = PB^2$$
$$\therefore\ AM^2 - MB^2 = PA^2 - PB^2 = k.$$
$$\text{or } x^2 - (a - x)^2 = k.$$
$$\text{and } x = \frac{a^2 + k}{2a}.$$

Hence **AM** is constant and **M** is a fixed point.

$\therefore$ the locus of **P** is a st. line $\perp$ **AB** drawn through the fixed point **M**.

21. **A, B are the centres of two circles of radii R, *r* respectively.**

To prove that the radical axis of the circles is a st. line $\perp$ AB and cutting it at a point M such that

$$AM^2 - MB^2 = R^2 - r^2.$$

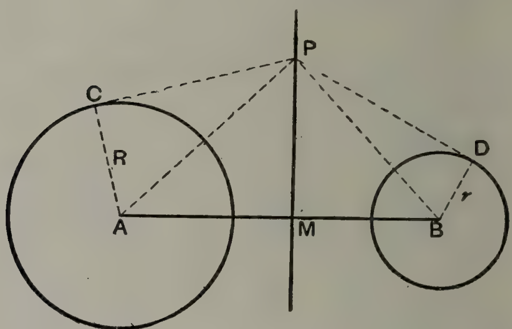

FIG. 26.

Let **P** be any point on the radical axis, and draw **PM** $\perp$ **AB**.

Draw the tangents **PC, PD** to the circles, and join **PA, PB, AC, BD.**

$$PA^2 = PC^2 + R^2,$$
$$PB^2 = PD^2 + r^2,$$

and, since **P** is on the radical axis, **PC = PD**;

$\therefore$ $PA^2 - PB^2 = R^2 - r^2$, a constant.

$\therefore$, by § 20, the locus of **P** is a st. line $\perp$ **AB**.

Also $PA^2 = AM^2 + PM^2$;

$PB^2 = MB^2 + PM^2$;

$\therefore$ $PA^2 - PB^2 = AM^2 - MB^2$;

and $\therefore$ the radical axis cuts **AB** at the fixed point **M**, such that $AM^2 - MB^2 = R^2 - r^2$.

22. To draw the radical axis of two non-intersecting circles.

First Method

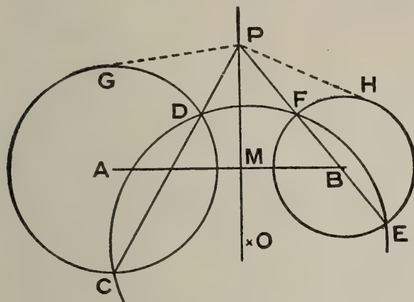

FIG. 27.

Let A, B be the centres of the circles.

Describe a circle with centre O cutting the given circles at C, D and E, F. Draw CD, EF and produce them to meet at P. Draw PM $\perp$ AB. Draw tangents PG, PH to the circles.

Show that PM is the required radical axis.

Second Method

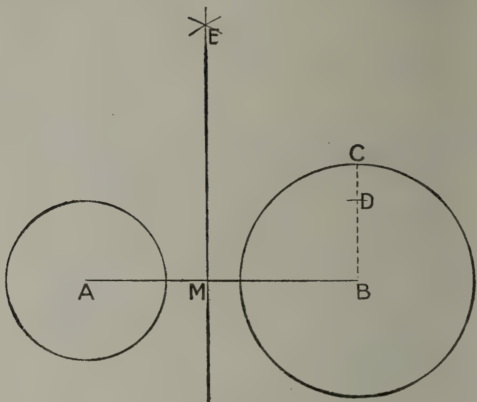

FIG. 28.

Let A, B be the centres of the two circles.

Join AB. Through B draw BC $\perp$ AB cutting the circle with centre B at C, and cut off BD equal to the radius of the other circle.

With centre A and radius AD describe an arc, and with centre B and radius AC describe another arc cutting the first at E. Draw EM $\perp$ AB.

$$BM^2 = BE^2 - EM^2 = AB^2 + BC^2 - EM^2.$$

$$MA^2 = AE^2 - EM^2 = AB^2 + BD^2 - EM^2.$$

$$\therefore\ BM^2 - MA^2 = BC^2 - BD^2.$$

$\therefore$, by § 21, EM is the radical axis.

23.—Exercises

1. Draw two circles, radii 1 inch and 2 inches, with their centres 4 inches apart. Find a point whose tangents to the two circles are each $1\frac{1}{2}$ inches in length.

2. The radical axis of two circles bisects their common tangents.

3. Find the radical axis of two circles which touch each other, internally or externally.

4. Prove that the radical axes of any three circles taken two and two together meet in a point.

NOTE.—This point is called the **radical centre** of the three circles.

5. **O** is a fixed point outside a given circle. **P** is any point such that the tangent from **P** to the given circle = **PO**. Show that the locus of **P** is a st. line $\perp$ to the line joining **O** to the centre of the circle.

6. **C** is a point on the circumference of a circle with centre **A**. Join **AC** and draw **CB** $\perp$ **CA**. With centre **B** and radius **BC** describe a circle.

   (*a*) The tangents to the circles at a common point are $\perp$ to each other.

   (*b*) The square on the line joining the centres of the circles equals the sum of the squares on their radii.

Definition.—Circles which cut each other so that the tangents at a common point are at right angles to each other are said to be **orthogonal.**

7. If **O** be the orthocentre of $\triangle$ **ABC**, the circles described on **AB** and **CO** as diameters are orthogonal.

8. If circles are described on the three sides of a $\triangle$ as diameters, their radical centre is the orthocentre of the $\triangle$.

9. Through two given points **A** and **B** draw any number of circles. What is the locus of their centres? Show that any two of this system of circles have the same st. line for radical axis.

Definition.—A system of circles that have the same radical axis are said to be **coaxial.**

10. To draw a system of circles coaxial with two given non-intersecting circles.

Let **A, B** be the centres of the given circles. Draw their radical axis **PO** cutting **AB** at **O.** From **O** draw a tangent **OE** to either circle. With centre **O** and radius **OE** describe a circle cutting **AB** at **C, D.** On the circle **CDE** take any point **F** and at **F** draw a tangent to the circle **CED** cutting **AB** at **G.** With centre **G** and radius **GF** describe a circle. Prove that this circle is coaxial with the given circles.

By taking different positions of **F** on the circle **CED** any number of circles may be drawn coaxial with the given circles.

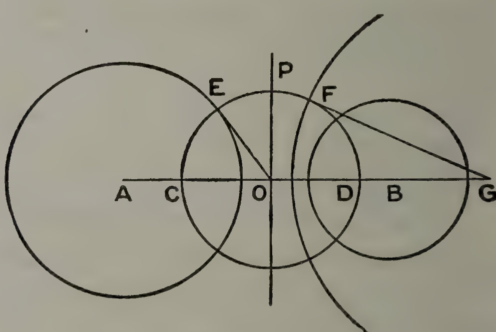

FIG. 29.

Definition.—No circle of the coaxial system has its centre between **C** and **D,** and consequently these points are called the **limiting points** of the system.

11. In Fig. 29, show that:—

(*a*) The circle **CED** cuts each circle of the coaxial system orthogonally;

(*b*) Any circle with centre in **PO** and passing through **C, D** cuts any circle of the coaxial system orthogonally.

(*b*)

12. If from any point **P** tangents be drawn to two circles, the difference between their squares equals twice the rectangle contained by the $\perp$ from **P** on the radical axis of the two circles and the distance between their centres.

13. The difference of the squares of the tangents drawn from a point to two fixed circles is constant. Show that the locus of the point is a st. line $\perp$ to the line of centres of the circles.

14. The tangent drawn from a limiting point to any circle of a coaxial system is bisected by the radical axis.

15. Show that the locus of the centre of a circle, the tangents to which from two given points are respectively equal to two given st. lines, is the radical axis of the circles having the given points as centres and radii respectively equal to the two given st. lines.

16. With a given radius describe a circle to cut two given circles orthogonally.

17. **XYZ** is the pedal $\triangle$ of $\triangle$ **ABC**; **YZ, BC** meet in **L**; **ZX, CA** meet in **M**; **XY, AB** meet in **N**. Show that **L, M, N**, are on the radical axis of the circumscribed and N.-P. circles of $\triangle$ **ABC**.

(Note.—$\triangle$s **MAZ, MXC** are easily shown to be similar.)

18. Describe a circle to cut three given circles orthogonally.

(Note.—Use Ex. 4 and Ex. 11 (*b*).)

CHAPTER II

Medial Section

24. When a straight line is divided into two parts such that the square on one part is equal to the rectangle contained by the given straight line and the other part, the straight line is said to be divided in **medial section.**

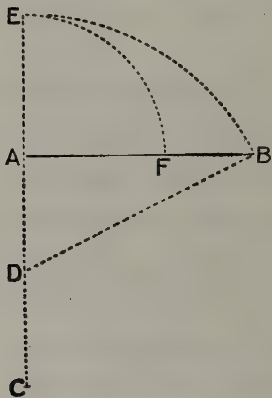

Fig. 30.

25. **To divide a given straight line internally in medial section.**

Let **AB** be the given st. line.

Draw **AC** $\perp$ **AB** and = **AB.** Bisect **AC** at **D.** With centre **D** and radius **DB** describe an arc cutting **CA** produced at **E.** With centre **A** and radius **AE** describe an arc cutting **AB** at **F.**

Then **AB** is divided in medial section at **F.**

$$
\begin{aligned}
DA^2 + AB^2 &= DB^2 \\
&= DE^2 \\
&= DA^2 + 2\,DA \,.\, AE + AE^2 \\
&= DA^2 + AB \,.\, AF + AF^2;
\end{aligned}
$$

$\because\ 2\,DA = AC = AB$ and $AE = AF.$

$$
\begin{aligned}
\therefore\ AF^2 &= AB^2 - AB \,.\, AF. \\
&= AB\,(AB - AF) \\
&= AB \,.\, BF.
\end{aligned}
$$

26. To divide a given straight line externally in medial section.

Let AB be the given st. line.

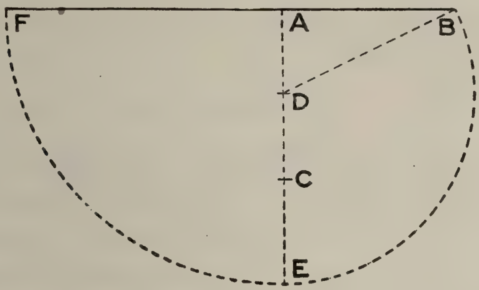

FIG. 31.

Draw $AC \perp AB$ and $= AB$. Bisect AC at D. With centre D and radius DB describe an arc cutting AC produced through C at E. With centre A and radius AE describe an arc cutting BA produced through A at F.

Then AB is divided externally at F such that $AF^2 = AB \cdot BF$.

$$\begin{aligned} DA^2 + AB^2 &= DB^2 \\ &= DE^2 = (AE - AD)^2 \\ &= AE^2 - 2\,AE \cdot AD + AD^2 \\ &= AF^2 - AF \cdot AB + AD^2 \\ \therefore \quad AF^2 &= AB^2 + AF \cdot AB \\ &= AB\,(AB + AF) \\ &= AB \cdot BF. \end{aligned}$$

27.—Exercises

1. If a st. line **AB** be divided at **F** so that $AF^2 = AB . BF$, show that $AB : AF = AF : BF$.

Give a general statement of this result.

2. **AB** is divided internally at **F** such that $AF^2 = AB . BF$. Show that $AF > FB$.

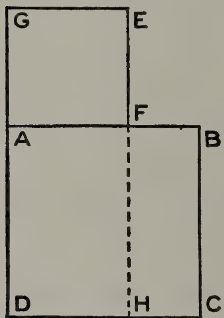

FIG. 32.

3. **AB** is divided in medial section at **F**. On **AB**, **AF** squares **ABCD**, **AFEG** are described as in Fig. 32. **EF** is produced to meet **DC** in **H**. Show that the rectangle **DE** = the square **DB**.

4. In Fig. 32 join **GB**, **DF**, produce **DF** to meet **GB**, and show that $DF \perp GB$.

5. A given st. line **AB** is to be divided in medial section. Let **F** be the point of section, a the length of **AB**, x the length of **AF**.

Then, by the definition of medial section, $x^2 = a\,(a - x)$

$$\text{or, } x^2 + ax - a^2 = 0.$$

Solving this quadratic equation, $x = \dfrac{-a \pm a\sqrt{5}}{2}$.

Show that the construction in § 25 is suggested by the root $\dfrac{-a + a\sqrt{5}}{2}$; and the construction in § 26 by the root $\dfrac{-a - a\sqrt{5}}{2}$.

6. Divide a st. line 4 inches in length in medial section. Measure the length of each part, and test the results by calculation.

7. The difference of the squares on the parts of a st. line divided in medial section equals the rectangle contained by the parts.

8. If AB be divided at C so that $AC^2 = AB \cdot BC$, show that $AB^2 + BC^2 = 3\ AC^2$.

9. If the sides of a rt.-∠d △ are in continued proportion, the ⊥ from the rt. ∠ divides the hypotenuse in medial section.

10. Describe a rt.-∠d △ whose sides are in geometrical progression.

---

28. **To describe an isosceles triangle having each of the angles at the base double the vertical angle.**

Draw a st. line AB and divide it at H so that $AH^2 = AB \cdot BH$. (§ 25.)

Describe arcs with centres A, B and radii AB, AH respectively, and let them cut at C.

Join AC, BC.

ABC is the required △.

Join HC.

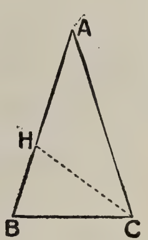

FIG. 33.

∠ B is common to the △s ABC, CBH, and since

$$\frac{AB}{AH} = \frac{AH}{BH}, \quad \therefore \frac{AB}{BC} = \frac{BC}{BH},$$

∴ these △s are similar and ∠ BCH = ∠ A.

$$\text{Again} \quad \frac{AC}{BC} = \frac{AB}{AH} = \frac{AH}{HB},$$

∴ CH bisects ∠ ACB.

∴ ∠ ACB = twice ∠ BCH = twice ∠ A;

and also ∠ ABC = twice ∠ A.

29.—Exercises

(*a*)

1. Express the ∠s of the △ **ABC** (Fig. 33) in degrees.

2. Construct ∠s of 36°, 18°, 9°, 6°, 3°.

3. Show that **AHC** (Fig. 33) is an isosceles △ having the vertical ∠ three times each of the base ∠s.

4. Show that each ∠ of a regular pentagon is 108°; and that, if a regular pentagon is inscribed in a circle, each side subtends at the centre an ∠ of 72°.

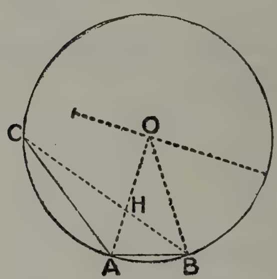

FIG. 34.

5. In a given circle **CAB** draw any radius **OA**. Divide **AO** at **H** so that $OH^2 = AO \cdot AH$. Place the chord **AB** = **HO**.

Join **BH** and produce **BH** to cut the circumference at **C**. Join **AC**.

Show that **AB** is a side of a regular decagon inscribed in the circle; and that **AC** is a side of a regular pentagon inscribed in the circle.

6. In a given circle inscribe a regular pentagon. At the angular points of the pentagon draw tangents meeting at **A, B, C, D, E.** Show that **ABCDE** is a regular pentagon circumscribed about the circle.

7. Show that each diagonal of a regular pentagon is ∥ to one of its sides.

8. Draw a regular pentagon on a given st. line.

9. **ABCDE** is a regular pentagon. Show that **AD, BD** trisect ∠ **CDE.**

10. **ABCDE** is a regular pentagon. Show that **AC, BD,** divide each other in medial section.

11. Construct a regular 5-pointed star. What is the measure of the $\angle$ at each vertex?

(*b*)

12. Show that the side of a regular decagon inscribed in a circle of radius r is $\frac{r}{2}(\sqrt{5}-1)$.

13. Show that the side of a regular pentagon inscribed in a circle of radius r is $\frac{r}{2}\sqrt{10-2\sqrt{5}}$.

14. The square on a side of a regular pentagon inscribed in a circle equals the sum of the squares on a side of the regular inscribed decagon and on the radius of the circle.

15. In a circle of radius 2 inches inscribe a regular decagon by the method of Ex. 5. Measure a side of the decagon and check your result by calculation.

16. In a circle of radius 3 inches inscribe a regular pentagon by the method of Ex. 5. Measure a side of the pentagon and check your result by calculation.

17. In a given circle draw two radii **OA, OB** at rt. $\angle$s to each other. Bisect **OB** at **C**. Join **AC**, and cut off **CD = CO.**

Show that **AD** is equal to a side of a regular decagon inscribed in the circle.

The regular inscribed pentagon may be drawn by joining alternate points obtained by placing successive chords each equal to **AD.**

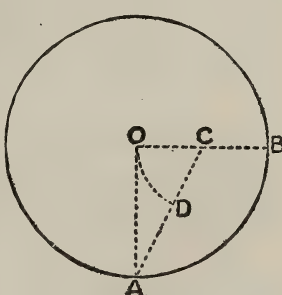

FIG. 35.

18. On a st. line 2 inches in length describe a regular pentagon. Measure a diagonal of the pentagon and check your result by calculation.

19. If the circumference of a circle be divided into n equal arcs,

(*a*) The points of division are the vertices of a regular polygon of n sides inscribed in the circle;

(*b*) If tangents be drawn to the circle at these points, these tangents are the sides of a regular polygon of n sides circumscribed about the circle.

20. Show that the difference between the squares on a diagonal and on a side of a regular pentagon is equal to the rectangle contained by them.

Miscellaneous Theorems

30. ABC is a triangle and P is a point in BC such that $\frac{BP}{PC} = \frac{n}{m}$. It is required to show that

$$mAB^2 + nAC^2 = (m+n)\ AP^2 + mBP^2 + nCP^2.$$

Draw AX $\perp$ BC.

From $\triangle$ ABP,

$AB^2 = AP^2 + BP^2 - 2\ BP . PX.$

From $\triangle$ APC,

$AC^2 = AP^2 + CP^2 + 2\ CP,\ PX.$

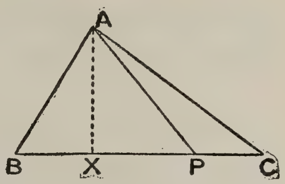

Fig. 36.

Multiplying both sides of the first of these equations by m, both sides of the second by n, adding the results and using the condition $mBP = nPC$, we obtain

$$mAB^2 + nAC^2 = (m+n)\ AP^2 + mBP^2 + nCP^2.$$

What does the result in § 30 become when $m = n$?

In a $\triangle$ ABC, $a = 77$ *mm*, $b = 90$ *mm* and $c = 123$ *mm*. Find the distances from C to the points of trisection of AB.

31. **In a right-angled triangle a rectilineal figure described on the hypotenuse equals the sum of the similar and similarly described figures on the other two sides.**

ABC is a $\triangle$ rt.-$\angle$d at C and having the similar and similarly described figures X, Y, Z on the sides.

It is required to show that $X + Y = Z$.

Similar figures are to each other as the squares on corresponding sides.

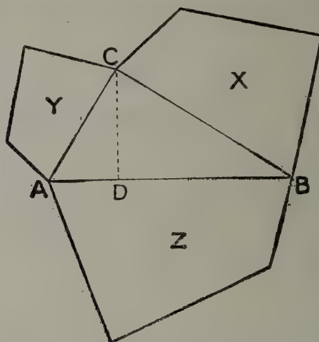

FIG. 37.

$$\therefore \frac{Y}{Z} = \frac{AC^2}{AB^2},$$

$$\text{and } \frac{X}{Z} = \frac{BC^2}{AB^2}.$$

$$\therefore \frac{X+Y}{Z} = \frac{AC^2+BC^2}{AB^2}.$$

But $AC^2 + BC^2 = AB^2$.

$$\therefore X + Y = Z.$$

Prove this theorem by drawing a $\perp$ from **C** to **AB** and using the theorem:—If three st. lines are in continued proportion, as the first is to the third so is any polygon on the first to the similar and similarly described polygon on the second.

Similar and Similarly Situated Polygons

32. Similar polygons are said to be **similarly situated** when their corresponding sides are parallel and drawn in the same direction from the corresponding vertices.

33. If two similar triangles have their corresponding sides parallel, the st. lines joining corresponding vertices are concurrent.

Let **ABC**, **DEF** be two similar $\triangle$s having the sides **BC**, **CA**, **AB** respectively $\parallel$ to the corresponding sides **EF**, **FD**, **DE**.

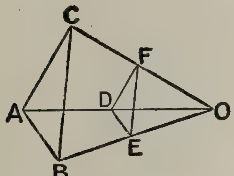

Fig. 38.

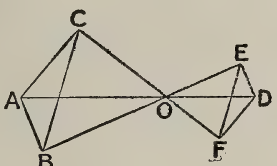

Fig. 39.

Prove **AD**, **BE**, **CF** concurrent.

34. When two similar polygons are so situated that their corresponding sides are parallel but drawn in opposite directions from the corresponding vertices, they are said to be **oppositely situated.**

In Fig. 39, the similar $\triangle$s **ABC**, **DEF** are oppositely situated.

35. If two similar polygons have the sides of one respectively parallel to the corresponding sides of the other, the straight lines joining corresponding vertices are concurrent.

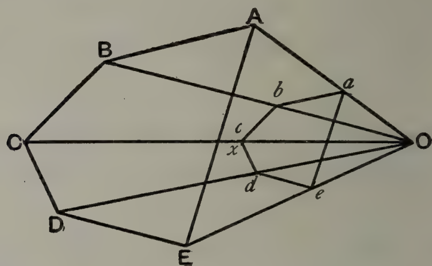

FIG. 40.

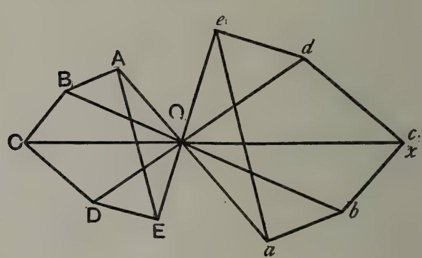

FIG. 41.

Let ABCDE, *abcde* be two similar polygons, similarly situated in Fig. 40, oppositely situated in Fig. 41.

Join Aa, Bb and let the joining lines meet at O. Join CO cutting bc at x.

From similar $\triangle$s,

$$\frac{ab}{AB} = \frac{Ob}{OB} = \frac{bx}{BC}.$$

But, by hypothesis,

$$\frac{ab}{\mathrm{AB}} = \frac{bc}{\mathrm{BC}}.$$

$$\therefore \quad bx = bc, \text{ and}$$

$\therefore$ OC passes through c.

Similarly it may be shown that the st. lines joining the remaining pairs of corresponding vertices pass through O.

36.—Exercises

(*a*)

1. Inscribe a square in a given △. Show that there are three solutions.

(Note.—In △ ABC on BC externally describe the square BDEC. Draw AD, AE cutting BC at F, G respectively. Draw FH, GK ⊥ BC meeting BA, CA at H, K respectively. Draw HK. Prove that HFGK is a square.)

2. In a given △ inscribe a rectangle similar to a given rectangle. Show that there are six solutions.

3. In a given semicircle inscribe a square.

4. In a given semicircle inscribe a rectangle having its sides in a given ratio.

5. In a given △ inscribe a △ having its sides ‖ to three given st. lines.

(Note.—From any point D in the side BC of the given △ ABC draw DE ‖ to one of the given st. lines and meeting AC at E. Draw DF, EF respectively ‖ to the other given lines. Draw CF cutting AB at G. Draw GH ‖ FD and GK ‖ FE, meeting BC, AC respectively at H, K. Draw HK. Show that GHK is the required △.)

(*b*)

6. The base of a square lies on one given st. line and one of its upper vertices lies on another given st. line. Show that the locus of the other upper vertex is a fixed st. line passing through the point of intersection of the two given lines.

7. In Figures 40 or 41 **P** is any point in **AB**, **Q** is any point in **CD** and p, q are the corresponding points in ab, cd respectively. Prove that **PQ** $\parallel$ pq.

8. $\triangle$ **ABC** $|||$ $\triangle$ abc, with their corresponding parts in the same circular order, but their corresponding sides are not $\parallel$. **BC**, bc meet at **P.** The circles **BP**b, **CP**c meet again at **O.** Prove that the $\triangle$ abc may be rotated about **O** to a position where it is similarly situated to $\triangle$ **ABC.**

(NOTE.—Prove $\angle$ **BO**b = $\angle$ **CO**c = $\angle$ **AO**a.)

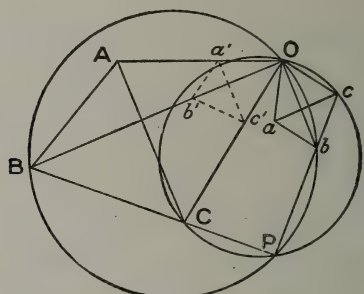

FIG. 42.

9. Show that $4\mathbf{R} = r_1 + r_2 + r_3 - r$, when **R** is the radius of the circumcircle, r of the inscribed circle, and r_1, r_2, r_3 the radii of the escribed circles.

(NOTE.—In $\triangle$ **ABC** let **I** be the centre of the inscribed circle, $\mathbf{I}_1$ that of the escribed circle opposite **A.** Draw the diameter **GDH** of the circumcircle bisecting **BC** at **D,** and meeting the circle above **BC** at **G** below at **H.** Draw **IM,** $\mathbf{I}_1\mathbf{N} \perp$ **BC** and produce **GH** to cut $\mathbf{MI}_1$ at **K.** Prove $2\ \mathbf{GD} = r_2 + r_3$ and $2\ \mathbf{DH} = r_1 - r$.)

10. If l, m, n are the $\perp$s from the circumcentre on the sides of a $\triangle$, show that

$$l + m + n = \mathbf{R} + r.$$

(NOTE.—Use the diagram and result of Ex. 9.)

CHAPTER III

Harmonic Ranges and Pencils

37. A set of collinear points is called a **range.**

38. A set of concurrent straight lines is called a **pencil.**

The lines are called the **rays of the pencil**; and their common point is called the **vertex of the pencil.**

39. When three magnitudes are such that the first has the same ratio to the third that the difference between the first and second has to the difference between the second and third, the differences being taken in the same order, the magnitudes are said to be in **harmonic proportion.** (H. P.)

Thus, if a, b and c represent three numbers such that $a : c = b - a : c - b$, a, b and c are in H. P.

40. **If in a range of four points A, C, B, D the st. line AB is divided internally at C and externally at D in the same ratio, the distances from either end of the range to the other three points are in H. P.**

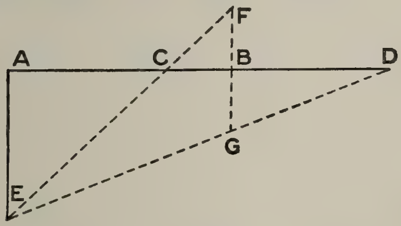

Fig. 43.

Draw ⊥s AE, FBG to AB, making FB = BG. Draw EF, EG cutting AB at C, D.

$$\text{Then } \frac{AC}{CB} = \frac{AE}{BF} = \frac{AE}{BG} = \frac{AD}{DB}.$$

(*a*) Then since $AC : CB = AD : DB$.

by alternation, $AC : AD = CB : DB$.

$\therefore \quad AC : AD = AB - AC : AD - AB$.

$\therefore$ by the definition of § 39, AC, AB, AD are in H. P.

(*b*) By inversion, $DB : AD = BC : AC$.

$\therefore \quad DB : DA = DC - DB : DA - DC$.

$\therefore$, by § 39, DB, DC, DA are in H. P.

State and prove a converse to this theorem.

41. When a range of four points A, C, B, D is such that $AC : CB = AD : DB$, it is called a **harmonic range.**

If any point P be joined to the four points of a harmonic range, the joining lines form a **harmonic pencil.**

42. **If, in the harmonic pencil P (A, C, B, D), a straight line through B parallel to PA cut PC, PD at E, F respectively, $BE = BF$.**

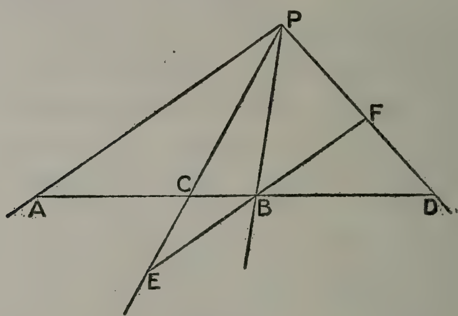

Fig. 44.

$$\because \quad \triangle ACP \;|||\; \triangle BCE,$$

$$\therefore \quad AP : EB = AC : CB.$$

$$\because \quad \triangle ADP \;|||\; \triangle BDF,$$

$$\therefore \quad AP : BF = AD : DB.$$

But, by hypothesis, $AC : CB = AD : DB$.

$$\therefore \quad AP : EB = AP : BF.$$

$$\therefore \quad EB = BF.$$

43. By a proof similar to that of § 42, the following converse to the theorem of that article may be shown to be true.

If, in the pencil P(A, C, B, D), a straight line through B parallel to PA cut PC, PD at E, F respectively such that BE = BF, then P(A, C, B, D) is a harmonic pencil.

44. **Any transversal is cut harmonically by the rays of a harmonic pencil.**

A transversal cuts the rays PA, PC, PB, PD of the harmonic pencil P(A, C, B, D) at K, M, N, L respectively.

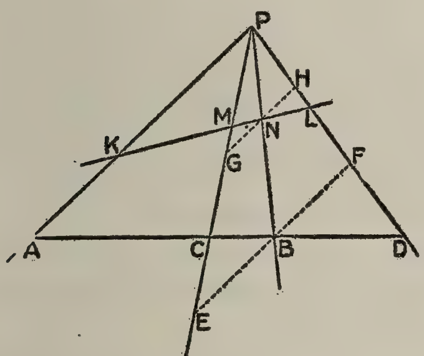

FIG. 45.

It is required to show that K, M, N, L is a harmonic range.

Through B, N respectively draw EF, GH ∥ PA.

∵ P(A, C, B, D) is a harmonic pencil, and EBF ∥ AP

∴, by § 42, EB = BF.

From similar △s, $\frac{GN}{NP} = \frac{EB}{BP}$ and $\frac{NP}{NH} = \frac{BP}{BF}$,

∴, by multiplication, GN : NH = EB : BF,

∴ GN = NH; and ∴, by § 43, K, M, N, L is a harmonic range.

45. If A, C, B, D is a harmonic range and O is the middle point of AB,

$$OB^2 = OC \cdot OD.$$

FIG. 46.

$$\because \quad \frac{AC}{CB} = \frac{AD}{DB};$$

$$\therefore \quad \frac{AC+CB}{AC-CB} = \frac{AD+DB}{AD-DB},$$

$$\text{or,} \quad \frac{2OB}{2OC} = \frac{2OD}{2OB},$$

$$\text{and } \therefore \quad OB^2 = OC \cdot OD.$$

State and prove a converse to this theorem.

46. If A, C, B, D is a harmonic range, A and B are said to be **harmonic conjugates** with respect to C and D; and C and D are said to be **harmonic conjugates** with respect to A and B.

47.—Exercises

(a)

1. Show how to find the fourth ray of a harmonic pencil when three rays are given.

2. Prove the theorem of § 44 when the transversal cuts the rays produced through the vertex.

3. In the △ ABC the bisectors of the interior and exterior ∠s at A cut BC and BC produced at D, E respectively. Show that B, D, C, E is a harmonic range.

4. A, C, B, D is a harmonic range and P is any point on the circle described on AB as diameter. Show that PA, PB respectively bisect the exterior and interior vertical ∠s of △ CPD.

(NOTE.—Draw EBF ∥ AP cutting PC, PD at E, F respectively.)

5. Using Ex. 4, give geometrical proofs of the converse theorems of § 45.

6. Two circles cut orthogonally. A st. line through the centre of either cuts the circles at **A, C, B, D.** Show that **A, C, B, D** is a harmonic range.

7. **A, C, B, D** is a harmonic range. Show that the circles described on **AB, CD** as diameters cut each other orthogonally.

(*b*)

8. **A, C, B, D** is a harmonic range; **O** is the middle point of **AB** and **R** is the middle point of **CD.** Show that:—

(*a*) $AB^2 + CD^2 = 4\ OR^2$;

(*b*) $CA \cdot CB = CD \cdot CO$;

(*c*) If **P** is any point on the range and lengths in opposite directions from **P** are different in sign, $PA \cdot PB + PC \cdot PD = 2\ PO \cdot PR$.

9. The inscribed circle of $\triangle$ **ABC** touches **BC, CA, AB** at **D, E, F** respectively, and **DF** meets **CA** produced at **P.** Show that **C, E, A, P** is a harmonic range.

(Note.—Use Menelaus' Theorem.)

10. The diameter **AB** of a circle is $\perp$ to a chord **CD.** **P** is any point on the circumference. **PC, PD** cut **AB,** or **AB** produced, at **E, F.** Show that **A, E, B, F** is a harmonic range.

11. Through **E,** the middle point of the side **AC** of the $\triangle$ **ABC,** a transversal is drawn to cut **AB** at **F, BC** produced at **D,** and a line through **B** $\parallel$ **CA** at **G.** Show that **G, F, E, D** is a harmonic range.

(Note.—Use § 43.)

12. A common tangent of two given circles is divided harmonically by any circle which is coaxial with the given circles.

(NOTE.—Use the converse of § 45.)

13. In a circle **AC**, **BD** are two diameters at rt. $\angle$s to each other, and **P** is any point on the quadrant **AD**. Show that **PA**, **PB**, **PC**, **PD** constitute a harmonic pencil.

14. In the harmonic pencil **O** (**A**, **C**, **B**, **D**), $\angle$ **AOC** = $\angle$ **COB** = $\angle$ **BOD**. Show that each of these $\angle$s = 45°.

15. A square is inscribed in a circle. Show that the pencil formed by joining any point on the circumference to the four vertices of the square is harmonic.

16. **P** is a fixed point and **OX**, **OY** two fixed st. lines. Show that the locus of the harmonic conjugate of **P** with respect to the points where any st. line drawn from **P** cuts **OX**, **OY** is a st. line passing through **O**.

The Complete Quadrilateral

48. The figure formed by four straight lines which meet in pairs in six points is called **a complete quadrilateral.**

The figure ABCDEF is a complete quadrilateral, of which AC, BD and EF are the three diagonals.

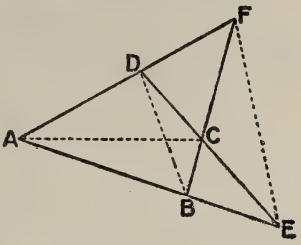

Fig. 47.

49. **In a complete quadrilateral each diagonal is divided harmonically by the two other diagonals, and the angular points through which it passes.**

ABCDEF is a complete quadrilateral having the diagonal AC cut by DB at P and by EF at Q.

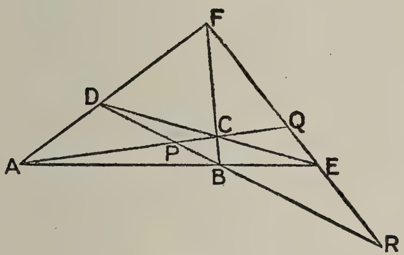

Fig. 48.

It is required to show that A, P, C, Q is a harmonic range.

In $\triangle$ ACF, AB, CD, FQ drawn from the vertices and meeting the opposite sides at B, D, Q are concurrent at E and $\therefore$, by Ceva's Theorem,

$$FD \cdot AQ \cdot CB = DA \cdot QC \cdot BF.$$

The transversal DPB cuts the sides of the $\triangle$ ACF at D, P, B and $\therefore$, by Menelaus' Theorem,

$$FD \cdot AP \cdot CB = DA \cdot PC \cdot BF.$$

Hence, by division,

$$\frac{AQ}{AP} = \frac{QC}{PC},$$

or, $$\frac{AP}{PC} = \frac{AQ}{QC},$$

and $\therefore$ A, P, C, Q, is a harmonic range.

From the above result it is seen that F (A, P, C, Q) is a harmonic pencil, and consequently, by § 44, D, P, B, R is a harmonic range.

Show, in the same manner that F, Q, E, R is a harmonic range.

Poles and Polars

50. If through a fixed point a line be drawn to cut a given circle and at the points of intersection tangents be drawn, the locus of the intersection of the tangents is called the **polar** of the fixed point; and the fixed point is called the **pole** of the locus.

51. **If C is the centre of a given circle and D is a fixed point, the polar of D with respect to the circle is a straight line which is perpendicular to CD and cuts it at a point E such that CE . CD equals the square on the radius.**

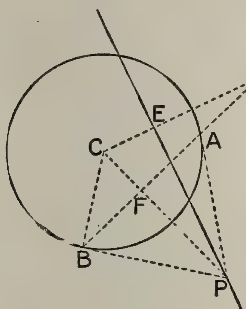

FIG. 49.

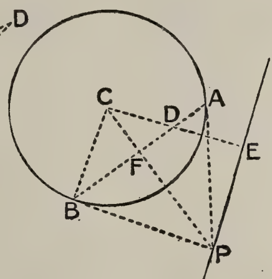

FIG. 50.

Through D draw any st. line cutting the circle at A and B. At A, B draw tangents to the circle intersecting at P.

P is a point on the polar of D.

Join CD and from P draw PE ⊥ CD.

Join CP cutting AB at F. Join CB.

∵ ∠s DEP, DFP are rt. ∠s,

∴ D, E, F, P are concyclic.

∴ CE . CD = CF . CP.

But CF . CP = CB^2.

∴ CE . CD = CB^2.

Then, since CD and CB are constants, CE must also be constant.

∴ the polar of D must be the st. line ⊥ CD through the fixed point E such that CE . CD = CB^2.

52. **If a point P lies on the polar of a point Q with respect to a circle, then Q lies on the polar of P.**

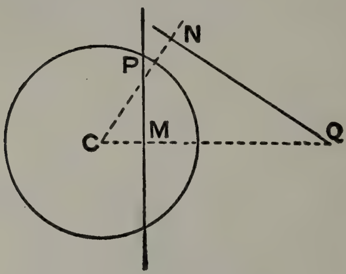

FIG. 51.

P is any point on PM the polar of Q.

To show that Q lies on the polar of P.

Join CP and draw QN $\perp$ CP.

$\because$ $\angle$s at M and N are rt. $\angle$s.

$\therefore$ Q, N, P, M are concyclic.

$\therefore$ CP . CN = CM . CQ but, by § 51, CM . CQ = the square on the radius.

$\therefore$ CP . CN = the square on the radius, and QN is the polar of P.

53. Two points, as P and Q in Fig. 51, such that the polar of each passes through the other, are called **conjugate points** with respect to the circle, and their polars are called **conjugate lines.**

A $\triangle$ such that each side is the polar of the opposite vertex is said to be **self-conjugate.**

54.—Exercises

(*a*)

1. **P** is a point at a distance of 4 cm. from the centre of a circle of radius 6 cm. Construct the polar of **P.**

2. **P** is a point at a distance of 7 cm. from the centre of a circle of radius 5 cm. Construct the polar of **P.**

3. Draw a st. line at a distance of 7 cm. from the centre of a circle of radius 4 cm. Construct the pole of the line.

4. When the point **P** is within the given circle, the polar of **P** falls without the circle; and when **P** is without the circle, the polar of **P** cuts the circle.

5. The polar of a point on the circumference is the tangent at that point.

6. **P** is a point without a given circle and the polar of **P** cuts the circle at **A.** Show that **PA** is a tangent to the circle.

Give a general statement of this theorem.

7. If any number of points are collinear, their polars with respect to any circle are concurrent.

8. Any number of lines pass through a given point; find the locus of their poles with respect to a given circle.

9. If the pole of a st. line **AB** with respect to a circle is on a st. line **CD,** the pole of **CD** is on **AB.**

10. The st. line joining any two points is the polar with respect to a given circle of the intersection of the polars of the two points.

11. The intersection of any two st. lines is the pole with respect to a given circle of the line joining the poles of the two st. lines.

12. Show how to draw any number of self-conjugate $\triangle$s with respect to a given circle.

(*b*)

13. If a st. line **PAB** cut a circle at **A**, **B** and cut the polar of **P** at **C**, and if **D** be the middle point of **AB**,

$$PA \cdot PB = PC \cdot PD.$$

14. Two circles **ABC**, **ABD** cut orthogonally. Show that the polar of **D**, any point on the circle **ABD**, with respect to the circle **ABC** passes through **E**, the point diametrically opposite to **D**.

15. The polar of any point **A** with respect to a given circle with centre **O** cuts **OA** at **B**. Show that any circle through **A** and **B** cuts the given circle orthogonally.

16. **A** is a given point and **B** any point on the polar of **A** with respect to a given circle. Show that the circle described on **AB** as diameter cuts the given circle orthogonally.

17. **ABC** is a △ inscribed in a circle, and a ‖ to **AC** through the pole of **AB** with respect to the circle meets **BC** at **D**. Show that **AD** = **CD**.

---

55. **Any straight line which passes through a fixed point is cut harmonically by the point, any circle, and the polar of the point with respect to the circle.**

P is the fixed point, **O** the centre of the circle, **PACB** any line through **P** cutting the circle at **A**, **B** and the polar **EC** of **P** with respect to the circle at **C**.

It is required to show that B, C, A, P is a harmonic range.

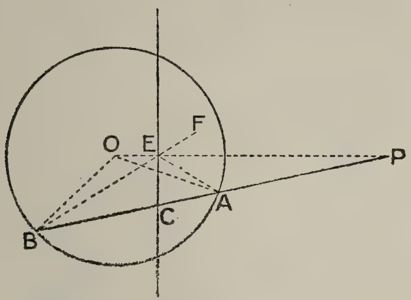

FIG. 52.

Join BO, OA, BE, EA and produce BE to F.

By § 51, OP . OE = OB^2, and ∴ OP : OB = OB : OE; also the ∠ BOE is common to the △ POB, BOE;

∴ these △ are similar,

and consequently ∠ OEB = ∠ OBP = ∠ OAB.

∴ the points B, O, E, A are concyclic.

∴ the ∠ AEP = ∠ OBA = ∠ OEB = ∠ FEP, and EP bisects the exterior vertical ∠ of △ EBA.

But ∠ PEC is a rt. ∠ and ∴ ∠ BEC = ∠ CEA.

$$\therefore \quad \frac{BC}{CA} = \frac{BE}{EA} = \frac{BP}{PA};$$

and B, C, A, P is a harmonic range.

If the point P is within the circle, C is without the circle, and by § 52, the polar of C passes through P.

∴ by the above proof B, P, A, C is a harmonic range.

Prove this theorem when the line PAB passes through the centre of the circle.

56. **ABCD** is a quadrilateral inscribed in a circle. **AB, DC** are produced to meet at **E**; **BC, AD** to meet at **F**, forming the complete quadrilateral **ABCDEF.**

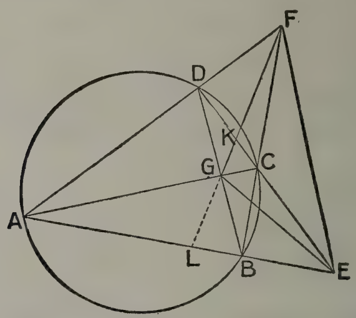

FIG. 53.

AC cuts **BD** at **G, FG** cuts **AB** at **L.**

From the complete quadrilateral **FDGCAB, A, L, B, E** and **D, K, C, E** are harmonic ranges. (§ 49.)

∴ **L** and **K** are points on the polar of **E**; (§ 55.) that is, **GF** is the polar of **E.**

Similarly, **GE** is the polar of **F.**

Hence **FE** is the polar of **G**; and the △ **EFG** is self-conjugate with respect to the circle **ABC.**

Cor.:—If from any point **E** two st. lines **EBA, ECD** are drawn to cut a circle at the points **B, A, C, D,** then the intersection of **BD** and **CA** is on the polar of **E,** and so also is the intersection of **BC** and **AD.**

57.—Exercises

(*a*)

1. Using a ruler only, find the polar of a given point with respect to a given circle.

2. Using a ruler only, draw the tangents from a given external point to a given circle.

3. Using a ruler only, find the pole of a given st. line with respect to a given circle.

4. **A** and **B** are two points such that the polar of either with respect to a circle, with centre **O**, passes through the other. Prove that the pole of **AB** is the orthocentre of the △ **AOB**.

5. Tangents **AB**, **AC** are drawn to a circle. The tangent at any point **P** cuts **BC**, **CA**, **AB** at **X**, **Y**, **Z** respectively. Show that **X**, **Z**, **P**, **Y** is a harmonic range.

6. If, in figure 53, **ABC** is a fixed △ and **D** is a variable point on the circle, prove that each side of the △ **EFG** passes through a fixed point.

(Note.—Use Ex. 9, § 54.)

7. **C** is the middle point of a chord **AB** of a circle, and **D**, **E** are two points on the circumference such that **CA** bisects the ∠ **DCE**. Prove that the tangents at **D** and **E** intersect on **AB**.

(Note.—At **C** draw the ⊥ to **AB** and produce it to meet **DE**.)

8. **D** is the middle point of the hypotenuse **BC** of the rt.-∠d △ **ABC**. A circle is described to touch **AD** at **A**. Prove that the polar of either of the points **B**, **C** with respect to the circle passes through the other point.

9. **A**, **B**, **C**, **D** are successive points on a st. line. Find points **X**, **Y** that are conjugate to each other both with respect to **A**, **B** and with respect to **C**, **D**.

(NOTE.—Draw a circle through **A**, **B** and another through **C**, **D**, intersecting each other at **P**, **Q**. Produce **PQ** to cut the given line at **O** and from **O** draw a tangent **OT** to either of the circles. With centre **O** and radius **OT** describe a circle cutting the given line at **X**, **Y**.)

10. **AD**, **BE**, **CF** are concurrent st. lines drawn from the vertices of △ **ABC** and cutting the opposite sides in **D**, **E**, **F**. **EF** meets **BC** at **X**. Show that **X** is the harmonic conjugate of **D** with respect to **B** and **C**.

11. A transversal cuts the sides **BC**, **CA**, **AB** of the △ **ABC** in **D**, **E**, **F**. The st. line joining **A** to the intersection of **BE** and **CF** meets **BC** at **H**. Show that **D** and **H** are harmonic conjugates with respect to **B** and **C**.

12. **P**, **Q**, are two conjugate points with respect to a circle. Show that the circle on **PQ** as diameter cuts the given circle orthogonally.

13. The ∠ **C** in △ **ABC** is obtuse and **O** is the orthocentre. A circle described on **OA** as diameter cuts **BC** at **D**. Show that the △ **ABC** is self-conjugate with respect to the circle with centre **O** and radius **OD**.

(*b*)

14. If a quadrilateral be circumscribed about a circle, the st. lines joining the points of contact of opposite sides are concurrent with the two diagonals of the quadrilateral.

(NOTE.—Produce the opposite chords of contact to meet, and use § 56—Cor.)

15. If **PM**, **QN** be respectively drawn ⊥ to the polars of **Q**, **P** with respect to a circle whose centre is **O**, **PM** : **QN** = **OP** : **OQ**. Salmon's Theorem.

(NOTE.—Draw **OK** ⊥ **PM**, **OL** ⊥ **QN**; and use similar △s **OPK**, **OQL**.)

16. **D**, **E**, **F** are points on the sides **BC**, **CA**, **AB** of the $\triangle$ **ABC**, such that **AD**, **BE**, **CF** are concurrent. Prove that the harmonic conjugates of **D** with respect to **B** and **C**, of **E** with respect to **C** and **A**, and of **F** with respect to **A** and **B** are collinear.

17. In $\triangle$ **ABC**, **X** is the projection of **A** on **BC**, and **BC** is produced to cut the radical axis of the circumcircle and the N.-P. circle at **P**. Show that **B**, **X**, **C**, **P** is harmonic.

18. The opposite sides of the quadrilateral **ABCD** are produced to meet at **E**, **F**. The diagonals of the complete quadrilateral form the $\triangle$ **LMN**. Show that the circle described on any one of the diagonals as diameter cuts the circumcircle of $\triangle$ **LMN** orthogonally.

Maxima and Minima

58. If a magnitude, such as the length of a st. line, an angle, or an area, varies continuously, subject to given conditions, it is said to be a **maximum** when it has its greatest possible value; and a **minimum** when it has its least possible value.

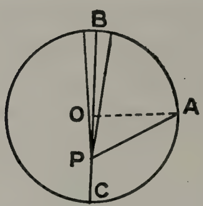

Fig. 54.

59. Let the distance, PA from a fixed point P within a circle to the circumference be the magnitude in question.

Join A to the centre O and produce PO to meet the circumference at B and C.

$$PA < PO + OA \text{ but} > OA - OP;$$

$$\therefore\ PA < PB \text{ but} > PC.$$

As PA rotates about P its length varies continuously. When it comes to the position PB it is greater than in the positions close to PB on either side, and has its **maximum** value.

Again, at PC it is less than in the positions close to PC on either side, and has its **minimum** value.

Draw the diagram and illustrate in the same manner the maximum and minimum distances from P to the circumference when P is without the circle.

What do the maximum and minimum values become when P is on the circumference?

Other simple examples are:—

(*a*) The $\perp$ is the minimum distance from a given point to a given st. line;

(*b*) The minimum distance between points on two $\parallel$ st. lines is $\perp$ to the $\parallel$ lines;

(*c*) The $\perp$ from a point on the circumference of a circle to a fixed chord is a maximum when the $\perp$, or the $\perp$ produced, passes through the centre.

60. (*a*) A and B are two fixed points on the same side of a fixed st. line CD. It is required to find the point P in CD such that PA + PB is a minimum.

Draw BM $\perp$ CD and produce making ME = BM. Draw AE cutting CD at P.

Then P is the required point.

Take any other point Q, in CD. Join PB, QA, QB, QE.

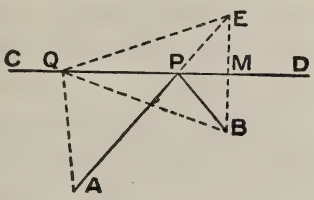

FIG. 55.

AQ + QE > AE. But QE = QB and PE = PB.

$$\therefore \text{AQ} + \text{QB} > \text{AP} + \text{PB};$$

and hence AP + PB is the minimum value.

It is easily seen that AP and PB make equal $\angle$s CPA and BPD with CD, and hence:—

The sum of the distances from A and B to the st. line is a minimum when the distances make equal $\angle$s with the line.

Find the point P when A and B are on opposite sides of AB.

(*b*) Of all $\triangle$s on the same base and having the same area the isosceles $\triangle$ has the least perimeter.

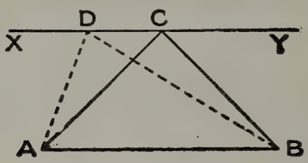

FIG. 56.

Since the area is constant the locus of the vertices is a st. line XY $\parallel$ AB.

If $\triangle$ ACB is isosceles, and DAB is any other $\triangle$ on AB and having its vertex in XY,

$$\angle XCA = \angle CAB = \angle CBA = \angle YCB;$$

$$\text{and } \therefore, \text{ by } (a),\ AC + CB < AD + DB;$$

that is, the perimeter of $\triangle$ ACB is less than that of any other $\triangle$ on the same base AB and having the same area.

(*c*) Of all $\triangle$s inscribed in a given acute-$\angle$d $\triangle$ the pedal $\triangle$ has the least perimeter.

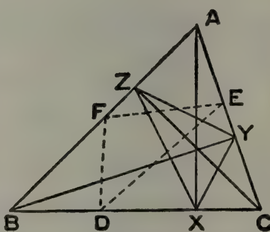

FIG. 57.

If DEF be any $\triangle$ inscribed in ABC and FE be considered fixed, by (*a*), the sum of FD and DE will be least when $\angle$ FDB = $\angle$ EDC; thus for the minimum perimeter the sides FD, DE must make equal $\angle$s with BC. Similarly DF, EF must make equal $\angle$s with AB and DE, EF must make equal $\angle$s with CA.

The sides of the pedal $\triangle$ XYZ make equal $\angle$s with the corresponding sides of $\triangle$ ABC.

$\therefore$ the perimeter of XYZ is less than that of any other $\triangle$ inscribed in ABC.

(*d*) The rectangle contained by the two segments of a st. line is a maximum when the st. line is bisected.

Let P be any point in AB, and O the middle point.

Describe the semicircle ACB and draw OC and PD $\perp$ AB. Join O, D.

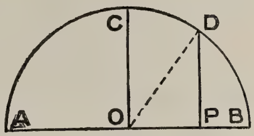

Fig. 58.

$$AP \,.\, PB = PD^2 \text{ and } AO \,.\, OB = OC^2;$$

$$\text{but } \because\ OC = OD \text{ and } OD > PD\ \therefore\ OC > PD;$$

$$\text{and } \therefore\ AO \,.\, OB > AP \,.\, PB.$$

(*e*) If the area of a rectangle is constant, its perimeter is a minimum when the rectangle is a square.

In Fig. 58, rect. $AP \,.\, PB = PD^2$.

The perimeter of the square $= 4\ PD$ while the perimeter of the rectangle $= 2\ AB = 4\ OD$, and $PD < OD$.

$\therefore$ The perimeter of the square is less than that of the rectangle of the same area.

61.—Exercises

(*a*)

1. Through a given point within a given circle draw the chord of min. length.

2. A and B are two fixed points, and CD is a fixed st. line. Find the point P in CD, such that the difference between PA and PB is a maximum.

(*a*) When A and B are on the same side of CD;

(*b*) When A and B are on opposite sides of CD.

3. Two sides AB, AC of a $\triangle$ are given in length. Show that the area of the $\triangle$ is a max. when $\angle$ A is a rt. $\angle$.

4. **A**, **B** are two fixed points and **P** is any point. Show that $PA^2 + PB^2$ is a min. when **P** is the middle point of **AB**.

5. **A**, **B**, **C** are fixed points and **P** is any point. Show that $PA^2 + PB^2 + PC^2$ is a min. when **P** is the centroid of the $\triangle$ **ABC**.

(NOTE.—See O.H.S. Geometry, Ex. 16, Page 133.)

6. Find the max. and min. distances between two points one on each of two given non-intersecting circles.

7. Given two adjacent sides describe the $\parallel$gm of max. area.

8. **A**, **B** are two fixed points. Find a point **P** on a fixed circle such that $PA^2 + PB^2$ is a max. or min.

9. Of all $\triangle$s of given base and given vertical $\angle$, the isosceles $\triangle$ has the greatest area.

10. Prove that the greatest rectangle that can be inscribed in a given circle is a square.

11. Give examples showing that if a magnitude vary continuously, there must be between any two equal values of the magnitude at least one maximum or minimum value.

12. Of all chords drawn through a given point within a circle, that which is bisected at the point cuts off the min. area.

13. From a given point without a circle, of which **O** is the centre, draw a st. line to cut the circumference in **L** and **M**, such that the $\triangle$ **OLM** may be a max.

(NOTE.—Use Ex. 3 in the analysis of this problem.)

14. Given two intersecting st. lines and a point within the $\angle$ formed by them, of all st. lines drawn through the point and terminated in the st. lines that which is bisected by it cuts off the min. area.

15. Given the base and the perimeter of a $\triangle$ show that the area is a max. when the $\triangle$ is isosceles.

16. Of all $\triangle$s having a given area, the equilateral has min. perimeter.

(NOTE.—Let **ABC** be a $\triangle$ having the given area and let two of the sides **AB**, **AC** be unequal. Then, by § 60, (*b*), if an isosceles $\triangle$ be described on **BC** of the same area, it will have less perimeter. $\therefore$ if any two of the sides be unequal the perimeter is not a min., and hence the equilateral $\triangle$ has the min. perimeter.)

17. Of all rt.-$\angle$d $\triangle$s on the same hypotenuse the isosceles $\triangle$ has the max. perimeter.

18. Find a point in a given st. line such that the sum of the squares of its distances from two given points is a min.

19. **A** and **B** are two given points on the same side of a given st. line; find the point in the line at which **AB** subtends the max. $\angle$.

(NOTE.—Describe a circle to pass through the two given points and touch the given st. line.)

20. Two towns are on opposite sides of a canal, unequally distant from it, and not opposite to each other. Where must a bridge be built, $\perp$ to the sides of the canal, that the distance between the towns, by way of the bridge, may be a min.?

(*b*)

21. A $\|$gm is inscribed in a given $\triangle$ by drawing from a point in the base st. lines $\|$ to the sides. Prove that the area of the $\|$gm is a max. when the lines are drawn from the middle point of the base.

22. The max. rectangle inscribed in a given $\triangle$ equals half the $\triangle$.

(NOTE.—Use Ex. 21.)

23. One circle is wholly within another circle, and contains the centre of the other. Find the max. and min. chords of the outer circle which touch the inner.

24. A, B are fixed points within a given circle. Find a point P on the circumference such that when PA, PB produced meet the circumference at C, D respectively, CD is a max.

(NOTE.—Describe a circle through A and B and touching the given circle.)

25. Find the point in a given st. line from which the tangent drawn to a given circle is a min.

26. Through a point of intersection A of two circles draw the max. st. line terminated in the two circumferences.

(NOTE.—Draw CAD $\parallel$ the line of centres and any other st. line EAF. Join C, D, E, F to the other point of intersection and use similar $\triangle$s.)

27. P, Q, R are points in the sides MN, NL, LM of $\triangle$ LMN and RQ $\parallel$ MN. Find the position of RQ for which the $\triangle$ PQR is a max.

(NOTE.—Use Ex. 21.)

28. A is a fixed point within the $\angle$ XOY. In OX, OY find points C, D respectively, such that the perimeter of the $\triangle$ ACD is a min.

29. A, B are points without a given circle. On the circle find points P and Q such that $\angle$ APB is a max. and $\angle$ AQB is a min.

30. The $\angle$ A of the $\triangle$ ABC is fixed and the sum of AB, AC is constant. Prove that BC is a min. when AB = AC.

31. A is a fixed point within the $\angle$ XOY. The st. line BAC cuts OX, OY at B, C. Prove that BA . AC is a min. when OB = OC.

32. From any point D in the hypotenuse BC of a rt.-$\angle$d $\triangle$ ABC $\perp$s DE, DF are drawn to AB, AC respectively. Find the position of D for which EF is a min.

33. If the sum of the squares on two lines is given, the sum of the lines is a max. when they are equal.

34. CAD is any st. line through a common point A of circles CAB, DAB. Prove that CA . AD is a max. when the tangents at C, D meet on BA produced.

(NOTE.—Let E, F be the centres of circles ABD, ABC respectively. Join EA, and draw the radius FC $\parallel$ EA. Join CA and produce to D. Then tangents at C, D will intersect on BA. Through A draw GAH terminated in the circles. Prove GA . AH $<$ CA . AD.)

35. Describe the maximum $\triangle$ DEF which is similar to a given $\triangle$ ABC and has its sides EF, FD, DE passing respectively through fixed points P, Q, R which are not collinear.

(NOTE.—On QR, RP describe segments containing $\angle$s = $\angle$ A, $\angle$ B respectively. Through R draw a st. line $\parallel$ to the line of centres of these segments and terminated in the arcs at D, E. DQ, EP meet at F, giving the max. $\triangle$ DEF. In the proof use the proposition that similar $\triangle$s are as the squares on homologous sides.)

36. A, B are fixed points on the same side of a fixed st. line XY. Place points P, Q on XY such that the distance PQ equals a given st. line and AP + BQ is a min.

(NOTE.—Through A draw AC $\parallel$ XY and equal to the given length for PQ. Draw CM $\perp$ XY and produce, making MD = CM. Join BD cutting XY at Q. Draw AP $\parallel$ CQ, cutting XY at P.)

Miscellaneous Exercises

62.—Exercises on Loci

(*a*)

1. Construct the locus of a point such that the ⊥s from it to two intersecting st. lines are in the ratio of two given st. lines.

2. A fixed point **O** is joined to any point **A** on a given st. line which does not pass through **O**. **P** is a point on **OA** such that the ratio of **OP** to **OA** is constant. Find the locus of **P**.

3. A fixed point **O** is joined to any point **A** on the circumference of a given circle, **P** is a point on **OA** such that the ratio of **OP** to **OA** is constant. Prove that the locus of **P** is a circle having its centre in the st. line joining **O** to the centre of the given circle.

Find the locus when **P** is on **AO** produced.

4. A fixed point **O** is joined to any point **A** on a given st. line which does not pass through **O**. **P** is a point on **OA** such that the rect. **OP . OA** is constant. Show that the locus of **P** is a circle.

Find the locus when **P** is on **AO** produced.

5. Through a fixed point **O** within an ∠ **YXZ** draw a st. line **MON**, terminated in the arms of the ∠, and such that the rect. **OM . ON** has a given area.

6. Find the locus of a point such that the sum of the squares on its distances from the arms of a given rt. ∠ is equal to the square on a given st. line.

7. The locus of a point, such that the sum of its distances from two given intersecting st. lines equals a given st. line, consists of the sides of a rectangle; and the locus of a point such that the difference of its distances from the intersecting st. lines equals the given st. line, consists of the produced parts of the sides of the rectangle.

8. Given the base **QR** of a $\triangle$ and the ratio of the other two sides, show that the locus of the vertex **P** is a circle with a diameter **ST** in the line **QR** such that **S, T** are harmonic conjugates with respect to **Q** and **R.**

(NOTE.—The circle of Apollonius, see O.H.S. Geometry, page 235.)

(HISTORICAL NOTE.—Apollonius of Perga died in Alexandria about 200 B.C.)

9. **AB** is a fixed chord in a circle and **C** is any point on the circumference. Show that the loci of the middle points of **CA, CB** are two equal circles.

10. Find the locus of the points from which tangents drawn to two concentric circles are $\perp$ to each other.

11. Construct the locus of the centre of the circle of given radius which intercepts a chord of fixed length on a given st. line.

12. Show that the locus of the centre of a circle of radius **R** which cuts a given circle at an $\angle$ **A** consists of two circles concentric with the given circle.

13. A circle rotates about a fixed point in its circumference. Show that the locus of the points of contact of tangents drawn $\parallel$ to a fixed st. line consists of the circumferences of two circles.

14. **AB, CD** are two chords of a circle, **AB** being fixed in position and **CD** of given length. Find the loci of the intersections of **AD, BC** and of **AC, BD.**

15. **A** and **B** are the centres of two circles which intersect at **C**; through **C** a st. line is drawn terminated in the circumferences at **D** and **E.** **DA, EB** are produced to meet at **P.** Find the locus of **P.**

16. In a quadrilateral **ABCD, AB** is fixed in position, **AC, BC** and **AD** are given in length:—

(*a*) Show that the locus of **P**, the middle point of **BD** is a circle having its centre at **E**, the middle point of **AB**, and its radius equal to half of **AD**;

(*b*) If **F** is the middle point of **AC**, show that the locus of the middle point of **FP** is a circle having its centre at the middle point of **FE** and its radius equal to one fourth of **AD**.

(*b*)

17. What is the locus of the point **P** when the st. line **MN** which joins the feet of the $\perp$s **PM**, **PN** drawn to two fixed lines **OX**, **OY** is of given length.

(NOTE.—Find **A** in **OY** such that **AR** drawn $\perp$ **OX** = **MN**. Draw the diameter **ME** of the circle through **O**, **M**, **N**, **P**; and join **NE**. $\triangle$ **MNE** $\equiv$ $\triangle$ **ORA**; $\therefore$ **ME** = **OA**. But **OP** = **ME**, $\therefore$ **OP** = **OA** and $\therefore$ the locus of **P** is a circle with centre **O** and radius **OA**.)

18. **BAC** is any chord passing through a fixed point **A** within a given circle with centre **E**. Circles described on **BA**, **AC** as chords touch the given circle internally at **B**, **C** respectively and cut each other at **D**. Show that the locus of **D** is a circle described on **AE** as diameter.

(NOTE.—**F**, **G** are centres of circles **BAD**, **CAD** respectively. Join **FG** cutting **EA** at **H**. Prove that **AFEG** is a $\|$gm and that **HD** = **HA**.)

19. Any secant **ABD** is drawn from a given point **A** to cut a given circle at **B** and **D**. Through **A**, **B** and **A**, **D** respectively two circles are drawn to touch the given circle; show that the locus of their second point of intersection is a circle on the line joining **A** to the centre of the given circle as diameter.

20. In $\triangle$ **ABC**, two circles touch **AB** at **B** and **AC** at **C** respectively and touch each other. Find the locus of their point of contact.

(NOTE.—Draw BD, CD $\perp$ respectively to BA, CA. Describe any circle with centre R in BD and passing through B. Produce DC to E making CE = RB. In DC find a point S equidistant from R and E. S is the centre of the circle which touches AC at C and the circle with centre R at P. Prove $\angle$ BPC $= 180° - \frac{A}{2}$ and that consequently the locus is a segment on BC.)

21. Any transversal cuts the sides BC, CA, AB of a given $\triangle$ ABC at D, E, F respectively. The circumscribed circles of the $\triangle$s AFE, CED cut again at P. Show that the locus of P is the circumcircle of $\triangle$ ABC.

22. From C, any point on the arc ACB, CD is drawn $\perp$ AB; with centre C and radius CD a circle is described. Tangents from A and B to this circle are produced to meet at P. Find the locus of P.

23. Two similar $\triangle$s ABC, AB′C′ have a common vertex A, and the $\triangle$ AB′C′ rotates in the common plane about the point A. Show that the locus of the point of intersection of CC′ and BB′ is the circumscribed circle of $\triangle$ ABC.

24. If a $\triangle$ ABC remains similar to itself while it turns in its plane about the fixed vertex A and the vertex B describes the circumference of a circle, show that the locus of C is a circle.

25. AB is a fixed diameter of a given circle, E the centre and C any point on the circumference. Produce BC to D making CD = BC. Show that the locus of the point of intersection of AC and ED is a circle on diameter AF such that A and F are harmonic conjugates with respect to E and B.

26. A rectangle inscribed in a given $\triangle$ ABC has one of its sides on BC. Show that the locus of the point of intersection of its diagonals is the line joining the middle point of BC to the middle point of the $\perp$ from A to BC.

(NOTE.—From the point where the median from **A** cuts the upper side of one of the rectangles draw a $\perp$ to **BC**.)

27. Any chord **BAC** is drawn through a fixed point **A** within a circle. On **BC** as hypotenuse a rt.-$\angle$d $\triangle$ **BPC** is described such that **A** is the projection of **P** on **BC**. Find the locus of **P**.

28. Any circle is drawn through the vertex of a given $\angle$. Show that the loci of the ends of that diameter which is $\parallel$ to the line joining the points where the circle cuts the arms of the $\angle$ are two fixed st. lines $\perp$ to each other and through the vertex of the given $\angle$.

29. Through **C**, a point of intersection of two given circles, a st. line **ACB** is drawn terminated in the circumferences at **A** and **B**. Prove that the locus of the middle point of **AB** is a circle passing through the points of intersection of the given circles, and having its centre at the middle point of the st. line joining their centres.

30. From a fixed point **P**, two st. lines **PA**, **PB**, at rt. $\angle$s to each other, are drawn to cut the circumference of a fixed circle at **A** and **B**. Show that the locus of the middle point of **AB** is a circle having its centre at the middle point of the st. line joining **P** to the centre of the given circle.

31. A $\parallel$gm is inscribed in a given quadrilateral **ABCD** with sides $\parallel$ **AC** and **BD**. The locus of the point of intersection of the diagonals of the $\parallel$gm is the st. line joining the middle points of the diagonals of the quadrilateral.

63.—THEOREMS

Definition.—If A, B be the centres of two circles, and points P, Q be found in AB and AB produced such that $\frac{AP}{PB} = \frac{AQ}{QB} = \frac{R}{r}$, the points P, Q are called the **centres of similitude** of the circles.

(*a*)

1. The centres of similitude of two circles are harmonic conjugates with respect to the centres of the circles.

2. Show that each of the four common tangents of two circles passes through one of the centres of similitude of the circles.

3. If || diameters be drawn in two circles, each of the four st. lines joining the ends of the diameters will pass through a centre of similitude of the circles.

4. If a circle touch two fixed circles, the line joining the points of contact passes through a centre of similitude of the two circles.

(NOTE.—Use Menelaus' Theorem.)

5. The six centres of similitude of three circles lie three by three on four st. lines.

6. If two circles cut orthogonally, any diameter of one which cuts the other is cut harmonically by that other.

7. In a system of coaxial circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.

8. Concurrent st. lines drawn from the vertices of the $\triangle$ ABC cut the opposite sides BC, CA, AB respectively at D, E, F. Show that

$$\sin ACF \,.\, \sin BAD \,.\, \sin CBE = \sin FCB \,.\, \sin DAC \,.\, \sin EBA.$$

9. Show that the area of $\triangle$ **ABC** $= \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$.

10. Find the area of the $\triangle$ **ABC** and also the radius of its circumcircle, given :—

(i) $a = 65$ mm., $b = 70$ mm., $c = 75$ mm.;

(ii) $a = 7$ cm., $b = 8$ cm., $c = 9$ cm.

11. If **L, M, N,** be the centres of the escribed circles of $\triangle$ **ABC,** the circumscribed circle of $\triangle$ **ABC** is the N.–P. circle of $\triangle$ **LMN.**

12. In Ex. 11 if **I** be the centre of the inscribed circle, **P** the point where the circumscribed circle cuts **IL** and **PH** be $\perp$ **AC, AH** equals half the sum and **CH** half the difference of b and c.

13. If **O** be the orthocentre of $\triangle$ **ABC, A, B, C, O** are the centres of the circles which touch the sides of the pedal $\triangle$.

14. **CA, CB** are two tangents to a circle; **E** is the foot of the $\perp$ from **B** on the diameter **AD**; prove that **CD** bisects **BE.**

(NOTE.—Produce **DB** to meet **AC** produced. Join **AB.**)

15. The $\perp$ from the vertex of the rt. $\angle$ on the hypotenuse of a rt.-$\angle$d $\triangle$ is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle.

(NOTE.—**AB** is the hypotenuse of the rt.-$\angle$d $\triangle$ **ABC,** p the length of the $\perp$ from **C** on **AB.** Then $s-a$, $s-b$ are the segments of **AB** made by the point of contact of the inscribed circle.

H.M. of segments $= \dfrac{2(s-a)(s-b)}{s-a+s-b} = \dfrac{(b+c-a)(c+a-b)}{2c} =$
$\dfrac{c^2-a^2-b^2+2ab}{2c} = \dfrac{ab}{c} = p.$)

16. The side of a square inscribed in a △ is half the harmonic mean between the base and the ⊥ from the vertex to the base.

17. The circumscribed centre of a △ is the orthocentre of the △ formed by joining the middle points of its sides; and the two △s have a common centroid.

18. **ABC** is a △. Describe a circle to touch **AC** at **C** and pass through **B**. Describe another circle to touch **BC** at **B** and pass through **A**. Let **P** be the second point of intersection of these circles. Show that ∠ **ACP** = ∠ **CBP** = ∠ **BAP**; and that the circumscribed circle of △ **APC** touches **BA** at **A**. Find another point **Q** such that ∠ **QBA** = ∠ **QAC** = ∠ **QCB**.

19. **O** is the orthocentre of △ **ABC, AX, BY, CZ** are the ⊥s from **A, B, C** on the opposite sides, **BD** is a diameter of the circumscribed circle. Show that :—

(*a*) **DC** = **AO**;

(*b*) $AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 =$ the square on the diameter of the circumscribed circle.

20. If a △ be formed with its sides equal to **AD, BE, CF**, the medians of △ **ABC**, the medians of the new △ will be respectively three-fourths of the corresponding sides of the original △.

(NOTE.—Draw **FG** ∥ and = **AD**. Join **CG**. Produce **FD, GD** to cut **CG, CF** at **K, H**.)

21. The opposite sides of a quadrilateral inscribed in a circle are produced to meet; show that the bisectors of the two ∠s so formed are ⊥ to each other.

22. **AG** is a median of the △ **ABC**. **BDEF** cuts **AG, AC** and the line through **A** ∥ **BC** at **D, E, F** respectively. Show that **B, D, E, F** is a harmonic range.

23. If A, C, B, D be a harmonic range, show that:—

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

24. Prove that the radical axis of the inscribed circle of △ ABC and the escribed circle which touches BC and AB, AC produced bisects BC.

25. If a st. line is divided in medial section and from the greater segment a part is cut off equal to the less, show that the greater segment is divided in medial section.

26. If a st. line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

27. In Figure 33, show that the centre of the circumcircle of △ HBC lies on the circumcircle of △ AHC.

28. Show that the radius of a circle inscribed in an equilateral △ is one-third of that of any one of the escribed circles.

29. A, B, C, D and P, Q, R, S are harmonic ranges and AP, BQ, CR are concurrent at a point O. Prove that DS passes through O.

30. A, B, C, D and A, E, F, G are harmonic ranges on two st. lines AD, AG. Prove that BE, CF, DG are concurrent.

(*b*)

31. Three circles pass through two given points P, Q. Two st. lines drawn from P cut the circumferences again at R, S, T and R′, S′, T′. Show that RS : ST = R′S′ : S′T′.

(Note.—Join the six points to Q.)

32. The middle points of the diagonals of a complete quadrilateral are collinear.

ABCDEF is a complete quadrilateral; L, M, N the middle points of its diagonals.

Draw LHK ∥ AE, produce KM to cut AB at G and join GH.

Prove L, M, N collinear.

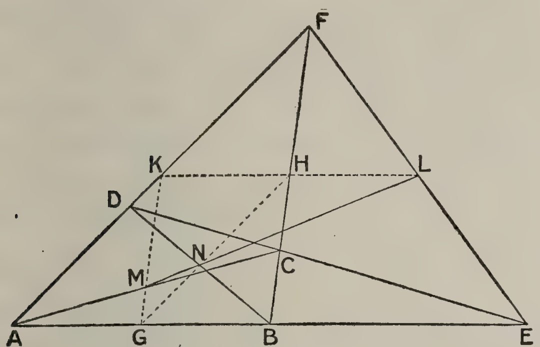

FIG. 59.

33. ABCD is a quadrilateral and O is a point within it such that △ AOB + △ COD = △ BOC + △ AOD; show that the locus of O is the st. line joining the middle points of the diagonals AC and BD.

(NOTE.—Produce DA. CB to meet at E. Make EF = AD and EG = BC. Join FO, EO, GO, FG.)

△ OEF = △ OAD, △ OEG = △ OBC. ∴ OFEG = half ABCD, and △ EFG is constant; ∴ △ OFG is constant and is on the constant base FG; ∴ the locus of O is a st. line.

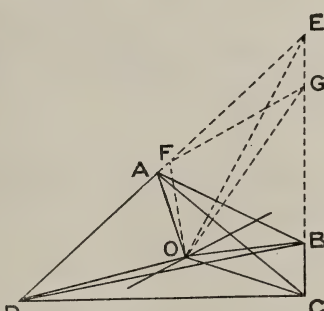

FIG. 60.

It is easily seen that the middle points of the diagonal are on the locus.

34. If a quadrilateral be circumscribed about a circle, the centre of the circle is in the st. line joining the middle points of the diagonals.

(NOTE.—Use Ex. 33.)

35. **G** is a fixed point in the base **BC** of the $\triangle$ **ABC** and **O** is a point within the $\triangle$ such that $\triangle$ **AOB** + $\triangle$ **COG** = $\triangle$ **AOC** + $\triangle$ **BOG**; show that the locus of **O** is the st. line joining the middle points of **BC** and **AG**.

(NOTE.—From **CB** cut off **CD** = **BG**. Join **OD, AD.**)

36. **G** is the point of contact of the inscribed circle of $\triangle$ **ABC** with **BC**. It is required to show that the centre of the circle is in the st. line joining the middle points of **BC** and **AG**.

37. **A** is a fixed point on a given circle and **P** is a variable point on the circle. **Q** is taken on **AP** produced so that **AQ : AP** is constant. Show that the locus of **Q** is a circle which touches the given circle at **A**.

38. **S** is a centre of similitude of two circles **PQT P′Q′T′**, and a variable line through **S** cuts the circles at the corresponding points **P, P′; Q, Q′**. Prove that, if **STT′** is a common tangent

$$SP \cdot SQ' = SP' \cdot SQ = ST \cdot ST'.$$

39. **AD** is a median of the $\triangle$ **ABC**. A st. line **CLM** cuts **AD** at **L** and **AB** at **M**. Prove that **ML : LC = AM : AB.**

40. The locus of the point at which two given circles subtend equal $\angle$s is the circle described on the join of their centres of similitude as diameter.

41. Having the N.–P. circle and one vertex of a $\triangle$ given, prove that the locus of its orthocentre is a circle.

(NOTE.—Let **A** be the given vertex and **K** the centre of the N.–P. circle. Join **AK** and produce to **M** making **KM = AK. M** is the centre of the locus.)

42. **AB** is a chord of a circle and the tangents at **A**, **B** meet at **C**. From any point **P** on the circle $\perp$s **PX**, **PY**, **PZ** are drawn to **BC**, **CA**, **AB** respectively. Prove that $PX \cdot PY = PZ^2$.

43. **OX**, **OY** are two fixed st. lines and from them equal successive segments are cut off; **AC**, **CE**, etc., on **OX**; **BD** **DF**, etc., on **OY**. Show that the middle points of **AB**, **CD**, **EF**, etc., lie on a st. line $\parallel$ to the bisector of the $\angle$ **XOY**.

(NOTE.—Produce the line joining the middle points of **AB**, **CD** to cut **OX**, **OY** and use Menelaus' Theorem.)

44. Show that the st. line joining the vertex of a $\triangle$ to the point of contact of the escribed circle with the base passes through that point of the inscribed circle which is farthest from the base.

(NOTE.—Show that the vertex and the point where the bisector of the vertical $\angle$ cuts the base are harmonic conjugates with respect to the inscribed and escribed centres; and use the resulting harmonic pencil having its vertex at the point of contact of the escribed circle with the base.)

From Ex. 44 show that the $\perp$ from the vertex to the base of a $\triangle$ and the bisector of the vertical $\angle$ are harmonic conjugates with respect to the lines joining the vertex to the points of contact of the inscribed and escribed circles with the base.

45. The five diagonals of a regular pentagon intersect at five points within it. Show that the area of the pentagon with these points for vertices is $\dfrac{7 - 3\sqrt{5}}{2}$ **A**, where **A** is the area of the given pentagon.

46. **ABCD** is a rectangle. If **A**, **P**, **C**, **Q** and **B**, **R**, **D**, **S** are each harmonic ranges, show that **P**, **Q**, **R**, **S** are concyclic.

47. If one pair of opposite sides of a cyclic quadrilateral when produced intersect at a fixed point, prove that the other pair when produced intersect on a fixed st. line.

What is the connection between the fixed point and the fixed st. line?

48. Concurrent st. lines drawn from the vertices of the $\triangle$ **ABC** cut the opposite sides **BC, CA, AB** respectively at **D, E, F.** Prove that the st. lines drawn through the middle points of **BC, CA, AB** respectively $\parallel$ to **AD, BE, CF** are concurrent.

49. The sides **AB, BC, CD, DA** of a quadrilateral touch a circle at **E, F, G, H** respectively. Show that the opposite vertices of **ABCD**, the intersection of the diagonals of **EFGH**, and the intersections of the opposite sides of **EFGH** form two sets of collinear points.

50. The circle **APQ** touches the circle **ABC** internally at **A.** The chord **BC** of the circle **ABC** is tangent to the circle **APQ** at **R**, and the chords **AB, AC** intersect the circle **APQ** in the points **P, Q.** Prove that **AP . RC = AQ . BR.**

51. Tangents to the circumcircle of $\triangle$ **ABC**, at the vertices, meet at **D, E, F. AD, BE, CF** are concurrent at **O.** Show that the $\perp$s from **O** on the sides of $\triangle$ **ABC** are proportional to the sides.

(NOTE.—Draw **GDH** $\parallel$ **FE** and meeting **AB, AC** produced at **GH.** Draw **DR, DS** $\perp$ **AG, AH.** Prove **DG = DH**; and, if **OM, ON** are $\perp$ **AB, AC**, that $\dfrac{OM}{ON} = \dfrac{DR}{DS} = \dfrac{AH}{AG} = \dfrac{AB}{AC}$.)

If **AD** cuts **BC** at **K**, show that $BK : KC = AB^2 : AC^2$.

52. **O** is the middle point of a chord **AB** of a circle, **DE, FG** are any chords through **O**; **EF, GD** cut **AB** at **H, K.** To prove **OK = OH.**

Produce EF, GD to meet at L; EG, FD at M. Produce ED to meet LM at P. Join OL, OM.

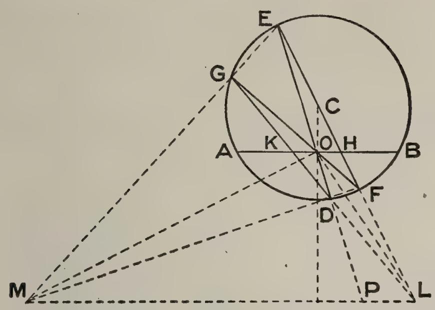

FIG. 61.

OLM is a self-conjugate $\triangle$; $\therefore$ CO produced cuts LM at rt. $\angle$s, and $\therefore$ AB $\parallel$ LM. PDOE is a harmonic range, and $\therefore$ L(P, D, O, E) is a harmonic pencil. Then $\because$ AB through O is $\parallel$ LM and cuts the other two rays at H, K; $\therefore$ OH = OK.

53. Through any point P in the median AD of $\triangle$ ABC a st. line is drawn cutting AB, AC at Q, R. Prove that PQ : PR = AC . AQ : AB . AR.

(NOTE.—Produce BP, CP to cut AC, AB at K, L. Join BR, CQ. BD . CL . PK = DC . LP . KB, $\therefore \dfrac{LP}{CL} = \dfrac{PK}{KB}$, $\therefore \dfrac{\triangle APQ}{\triangle ACQ} = \dfrac{\triangle APR}{\triangle ABR}$; or $\dfrac{\triangle APQ}{\triangle APR} = \dfrac{\triangle ACQ}{\triangle ABR}$ and $\therefore$ PQ : PR = AC . AQ : AB . AR.)

64.—Problems

(*a*)

1. Draw a st. line, terminated in the circumferences of two given circles, equal in length to a given st. line, and $\parallel$ to a given st. line.

2. Through a given point on the circumference of a circle draw a chord which shall be bisected by a given chord.

3. In the hypotenuse of a rt.-$\angle$d $\triangle$ find a point such that the sum of the $\perp$s on the arms of the rt. $\angle$ equals a given st. line. What are the limits to the length of the given st. line?

4. In the hypotenuse of a rt.-$\angle$d $\triangle$ find a point such that the difference of the $\perp$s on the arms of the rt. $\angle$ equals a given st. line.

When will there be two, one or no solutions?

5. In the hypotenuse of a rt.-$\angle$d $\triangle$ find a point such that the $\perp$s on the arms of the rt. $\angle$ are in a given ratio.

6. Through a given point draw a st. line terminated in the circumferences of two given circles and divided at the given point in a given ratio.

(Analysis.—Let **A** be the given point and suppose **PAQ** to be the required st. line terminated at **P**, **Q** in the circles of which the centres are respectively **C**, **D**. Join **CP**, and draw **QR** $\parallel$ **CP** meeting **CA** in **R**. From similar $\triangle$s, $\frac{CA}{AR} = \frac{CP}{QR}$ = the given ratio $\frac{PA}{AQ}$, wherein **CA**, **CP** are known; and $\therefore$ the position of **R** and the length of **QR** are known.)

7. In a given circle inscribe a rectangle having its perimeter equal to a given st. line.

8. In a given circle inscribe a rectangle having the difference between adjacent sides equal to a given st. line.

9. In a given circle inscribe a rectangle having its sides in a given ratio.

10. In a circle of radius 5 cm. inscribe a rectangle having its area 22 sq. cm.

(ANALYSIS.—Let x, y be the sides of the rectangle, then $xy = 22$ and $x^2 + y^2 = 100$. Show that $x + y = 12$.)

11. **A** and **B** are fixed points on the circumference of a given circle. Find a point **C** on the circumference such that **CA, CB** intercept a given length on a fixed chord.

(ANALYSIS.—Draw **BL** $\parallel$ to the given chord and equal to the given length. Join **L** to the point where **AC** is supposed to cut the given chord. Join **AL**, etc.)

12. **A** and **B** are fixed points on a circumference. Find a point **C** on the circumference such that **CA, CB** cut a fixed diameter at points equally distant from the centre.

(ANALYSIS.—Draw the diameter **AOD.** Join **D** to the point **F** where **BC** is supposed to cut the given diameter. Prove that the circumcircle of $\triangle$ **DFB** touches the given st. line **AD** at **D.**)

13. In a given circle inscribe a $\triangle$, such that two of its sides pass through given points, and the third side is a maximum.

14. Two towns are on different sides of a straight canal, at unequal distances from it, and not opposite to each other. Where must a bridge be built $\perp$ to the direction of the canal so that the towns may be equally distant from the bridge?

15. Divide a given st. line into two parts so that the squares on the two parts are in the ratio of two given st. lines.

16. Construct the locus of a point the difference of the squares of whose distances from two points 3 inches apart is 5 sq. inches.

17. Two points **A** and **B** are four inches apart. Construct the locus of the point the sum of the squares of whose distances from **A** and **B** is 20·5 square inches.

18. Divide a given st. line into two parts such that the sum of the squares on the whole st. line and on one part is twice the square on the other part.

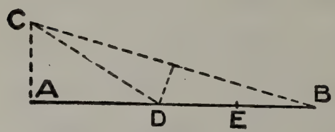

FIG. 62.

(ANALYSIS.—Let AB $(= a)$ be the required line and EB $(= x)$ a segment such that $a^2 + x^2 = 2(a - x)^2$. Then $x = 2a - a\sqrt{3}$, and $a - x = a\sqrt{3} - a$; $\therefore \frac{\text{AE}}{\text{EB}} = \frac{\sqrt{3}-1}{2-\sqrt{3}} = \sqrt{3}+1$. $\therefore$ AE = EB $\sqrt{3}$ + EB. Cut off ED = EB and draw AC $\perp$ AB and = EB. Then since AD = EB $\sqrt{3}$ = AC $\sqrt{3}$, $\angle$ ADC = 30° and CD = 2AC = DB. $\therefore$ $\angle$ B = 15°.

The following construction is then evident:—At B make $\angle$ ABC = 15°, and draw AC $\perp$ AB. Draw the rt.-bisector of BC cutting AB at D. Bisect DB at E.)

19. Two non-intersecting circles have their centres at A and B, and C is a point in AB. Draw a circle through the point C and coaxial with the two given circles.

(NOTE.—From O, the point where the radical axis cuts AB draw a tangent, OT, to one of the given circles. Then if CC′ is the diameter of the required circle, OC . OC′ = OT^2 and $\therefore$ OC′ is the third proportional to OC and OT.)

20. Construct a $\triangle$ having one side and two medians equal to three given st. lines. (Two cases.)

21. Construct a $\triangle$ having the three medians equal to three given st. lines.

22. Given the vertical $\angle$, the ratio of the sides containing it, and the diameter of the circumscribing circle; construct the $\triangle$.

23. Given the feet of the $\perp$s drawn from the vertices of a $\triangle$ to the opposite sides; construct the $\triangle$.

24. Draw a circle to touch a given circle, and also to touch a given st. line at a given point.

25. Draw a circle to pass through two given points and touch a given circle.

26. Draw a circle to pass through a given point and touch two given intersecting st. lines.

27. **AB** is the chord of a given segment of a circle. Find a point **P** on the arc such that **AP** + **BP** is a maximum.

28. Find a point **O**, within a △ **ABC** such that:—

(1) △ **AOB** : △ **BOC** : △ **COA** = 1 : 2 : 3;

(2) △ **AOB** : △ **BOC** : △ **COA** = l : m : n.

(*b*)

29. Find a point such that its distances from the three sides of a △ may be proportional to three given st. lines.

(Note.—Draw **BD**, **CE** ⊥ **BC**, and each = l. Join **DE**. Draw **AF** ⊥ **AC**, and = m. Draw **FG** ∥ **AC**, meeting **DE** at **G**. Draw **AH** ⊥ **AB**, and = n. Draw **HK** ∥ **AB**, meeting **DE** at **K**. **BK**, **CG** meet at **P** the required point. Show that, if **PL**, **PM**, **PN** be ⊥ to the sides, **PL** : **PM** : **PN** = l : m : n.)

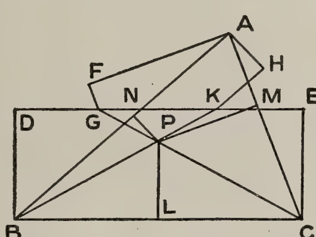

Fig. 63.

30. Through a given point within a circle draw a chord which shall be divided in a given ratio at the given point.

31. **A**, **B**, **C**, **D** are points in a st. line. Find a point at which **AB**, **BC**, **CD** subtend equal ∠s.

(Note.—The required point is the intersection of two circles of Apollonius. See O.H.S. Geometry, page 235.)

32. Given a vertex, the orthocentre and the centre of the N.–P. circle of a △, construct the △.

33. Having divided a st. line internally in medial section, find the point of external division in medial section by ratio and proportion.

34. Describe an equilateral △ with one vertex at a given point and the other two vertices on two given || st. lines.

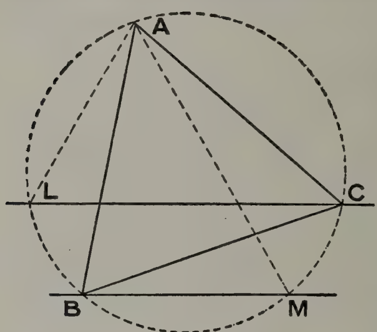

FIG. 64.

(ANALYSIS.—Describe a circle about the equilateral △ **ABC** cutting the || lines again at **L, M.** Then ∠ **ALC** = **ABC** = 60°, and ∠ **AMB** = ∠ **ACB** = 60°.)

35. Find the locus of the middle point of the chord of contact of tangents drawn from a point on a given st. line to a given circle.

36. The locus of the centre of a circle which bisects the circumferences of two given circles is a st. line ⊥ to the line of centres and at the same distance from the centre of one circle that the radical axis is from the centre of the other.

37. Describe a circle to bisect the circumferences of three given circles.

38. Find the locus of the centre of a circle that passes through a given point and also bisects the circumference of a given circle.

39. Describe a circle to pass through two given points and bisect the circumference of a given circle.

40. Given a point and a st. line, construct the circle to which they are pole and polar, and which passes through a given point.

Part II.—ANALYTICAL GEOMETRY

FORMULÆ

The following important results from algebra and trigonometry are frequently used in analytical geometry :—

1. The roots of the quadratic equations $ax^2 + 2bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - ac}}{a} \text{ and } \frac{-b - \sqrt{b^2 - ac}}{a}.$$

These roots are

real,	if $b^2 >$, or $=$, ac ;
imaginary,	if $b^2 < ac$;
equal to each other,	if $b^2 = ac$;
equal in magnitude but opposite in sign,	if $b = 0$;
rational,	if $b^2 - ac$ is a perfect square.
One root $= 0$,	if $c = 0$;
both roots $= 0$,	if $b = c = 0$.

The sum of the roots $= \dfrac{-2b}{a}$.

The product of the roots $= \dfrac{c}{a}$.

2. The fraction $\dfrac{a}{b} = \infty$, if $b = 0$ and a is not $= 0$.

3. The equation $ax + by + c = 0$ is the same as $px + qy + r = 0$,

$$\text{if } \frac{a}{p} = \frac{b}{q} = \frac{c}{r}.$$

4. $ax^2 + 2bx + c$ is a perfect square, if $b^2 = ac$.

5. If $a_1x + b_1y + c_1z = 0$

and $a_2x + b_2y + c_2z = 0$,

then $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$$

6. For all values of a,

$$\sin^2 a + \cos^2 a = 1.$$

7. If $\tan a = k$, $a = \tan^{-1} k$.

8. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$

9. $\sin A + \sin B = 2 \sin \frac{A+B}{2}, \cos \frac{A-B}{2}$, etc.

10. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$

CONTENTS

ELEMENTARY ANALYTICAL GEOMETRY

CHAPTER I

Cartesian Coordinates

1. **Analytical, or algebraic, geometry** was invented by Descartes in 1637, and this invention marks the beginning of the history of the modern period of mathematics. It differs from pure geometry in that it lays down a general method, in which, by a few simple rules, any property can be at once proved or disproved, while in the latter each problem requires a special method of its own.

2. **The Origin.** In plane analytical geometry the positions of all points in the plane are determined by their distances and directions as measured from a fixed point.

If the points are all in a st. line, the fixed point is most conveniently taken in that line.

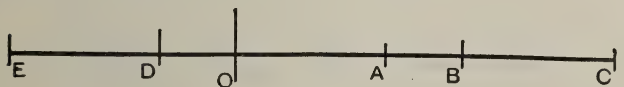

Fig. 1.

Thus, if the distance and direction of each of the points A, B, C, D, E from the point O are given, the positions of these points are known.

The point O is called the **origin**, or **pole**.

3. **Use of plus and minus.** In algebra the signs plus and minus are used to indicate opposite qualities of the numbers to which they are prefixed; and in analytical geometry, as in trigonometry, these signs are used to show difference of direction. In a horizontal st. line distances measured from the origin to the right are taken to be positive, while those to the left are negative; and in a vertical st. line distances measured upward are positive, while those measured downward are negative. Thus, in Fig. 1, if OA = 2 cm., OB = 3 cm., OC = 5 cm., OD = 1 cm., and OE = 3 cm., the positions of these points are respectively represented by 2, 3, 5, − 1 and − 3, the understood unit being one centimetre.

Rectangular Coordinates

4. **Coordinates.** When points are not in the same st. line, their positions are determined by their distances

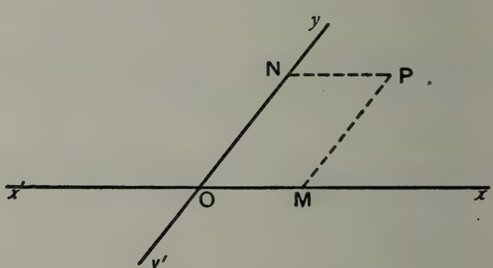

Fig. 2.

from two st. lines x'Ox and y'Oy drawn through the origin, the distances being measured in directions || to the given st. lines.

These lines are called the **axes of coordinates,** or shortly, the **axes.**

$x'Ox$ is called the axis of x, and $y'Oy$ is called the axis of y.

From a point P draw PM $\parallel Oy$ and PN $\parallel Ox$, terminated in the axes.

PM is called the **ordinate** of P, and PN (= OM), is called the **abscissa** of P. These two distances, the abscissa and ordinate, are called the **coordinates** of the point.

Sometimes, from the name of the inventor, they are spoken of as **cartesian coordinates.**

5. **Rectangular coordinates.** When the axes are at rt. $\angle$s to each other, the distances of a point from the axes are called its **rectangular coordinates.**

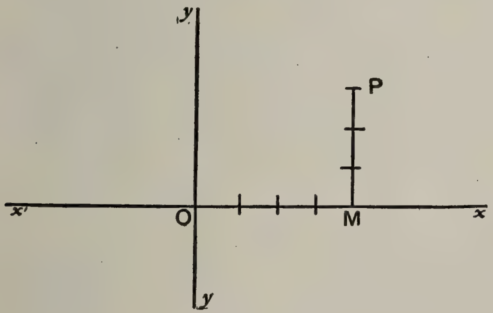

FIG. 3.

To locate the point of which the abscissa is 4 and the ordinate 3 when the coordinates are rectangular,

measure the distance OM = 4 units along Ox and at M erect the ⊥ PM = 3 units. P is the required point.

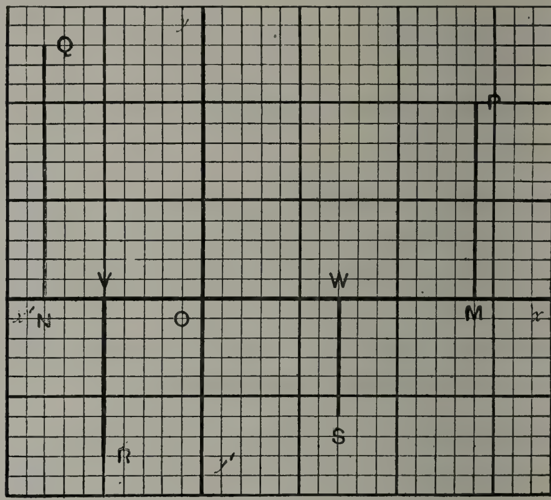

FIG. 4. (Unit = ½ inch.)

In Fig. 4, the abscissa of P = OM = 2·8, the ordinate of P = PM = 2. The position of this point is then indicated by the notation (2·8, 2). For Q, the abscissa = ON = - 1·6, the ordinate = QN = 2·6 and the position of the point is indicated by (−1·6, 2·6).

Similarly the position of R is (−1, −1·6), and that of S is (1·4, −1·2).

xOy, yOx', x'Oy' and y'Ox are respectively called the first, second, third and fourth quadrants; and we see from the diagram, that:—

for a point in the first quadrant both coordinates are positive;

for a point in the second quadrant the abscissa is negative and the ordinate is positive;

for a point in the third quadrant both are negative; and

for a point in the fourth the abscissa is positive and the ordinate is negative.

Thus the signs of the coordinates show at once in which quadrant the point is located.

6.—Exercises

1. Write down the coordinates of the points **A, B, C, D, E, F, G, H** and **O** in Fig. 5.

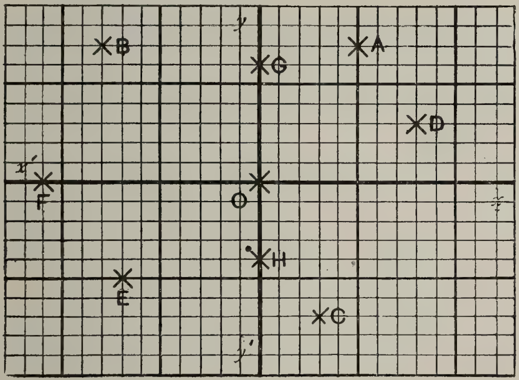

Fig. 5. (Unit = $\frac{1}{10}$ inch.)

2. Draw a diagram on squared paper, and mark on it the following points:—**A** (4, 3), **B** (4·6, 0), **C** (– 2, – 3), **D** (– 4, 2), **E** (0, 2·8). Indicate the unit of measurement on the diagram.

3. Draw a diagram on squared paper and mark the following points:—(4, 3), (3, 4), (– 3, – 4), (– 4, 3), (0, 5), (0, – 5), (– 5, 0). Describe a circle with centre **O** and

radius 5. Should the circle pass through the seven points? Why?

4. The side of an equilateral $\triangle = 2a$. One vertex is at the origin, one side is on the axis of x and the $\triangle$ is in the first quadrant. What are the coordinates of the three vertices?

5. One corner of a square is taken as origin and the axes coincide with two sides. The length of a side is b. What are the coordinates of the corners, the square being in the first quadrant?

THE DISTANCE BETWEEN TWO POINTS

7. In general, the abscissa of a point is represented by x, the ordinate by y.

8. **To find the distance between a point P (x_1, y_1) and the origin.**

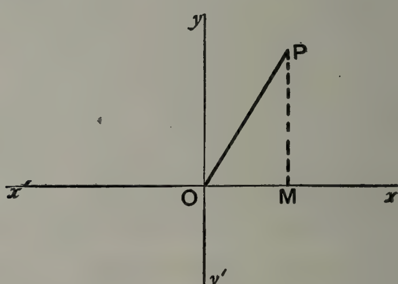

FIG. 6.

From P draw PM $\perp$ Ox.

$$\because \text{PMO is a rt.-}\angle\text{d } \triangle,$$

$$\therefore \text{PO}^2 = \text{OM}^2 + \text{PM}^2$$

$$= x_1^2 + y_1^2.$$

$$\therefore \text{PO} = \sqrt{x_1^2 + y_1^2}.$$

9. **To find the distance between** P (x_1, y_1) **and** Q (x_2, y_2).

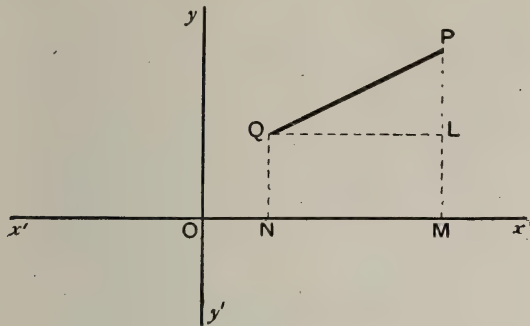

FIG. 7.

Draw PM and QN $\perp$ Ox; QL $\perp$ PM.

QL = NM = OM − ON = $x_1 - x_2$.

PL = PM − LM = PM − QN = $y_1 - y_2$.

$\because$ PLQ is a rt.-$\angle$d $\triangle$,

$\therefore$ PQ² = QL² + PL².

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

$$\therefore \text{PQ} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

10. If the point Q in § 9 coincides with the origin O, $x_2 = 0$ and $y_2 = 0$. Substituting these values, in the expression for PQ in that article we obtain

$$\text{PO} = \sqrt{x_1^2 + y_1^2}.$$

This shows that the result in § 8 is a particular case of that in § 9.

11. The result in § 9 holds good, *in the same form*, for any two points whether the coordinates are positive or negative.

For example—it is required to find the distance between P $(-3, 2)$ and Q $(5, -2)$.

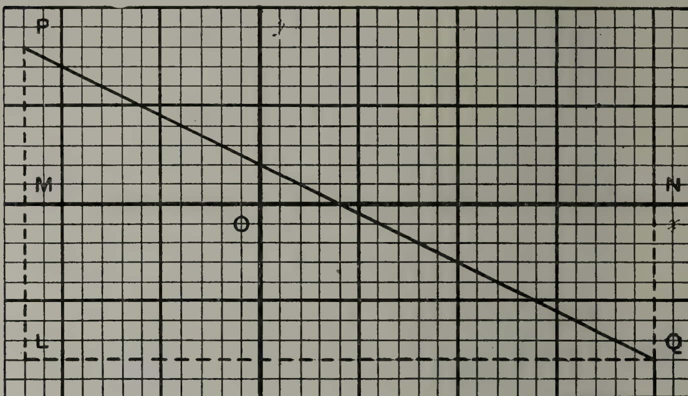

FIG. 8. (Unit = $\frac{4}{10}$ inch.)

Draw PM, QN $\perp$ Ox; QL $\perp$ PM.

The length of ML = length of NQ = 2.

$$\therefore \quad PL = PM + ML = 2 + 2 = 4.$$

The length of QL = NM = $5 + 3 = 8$.

$$PQ^2 = QL^2 + PL^2$$
$$= 64 + 16 = 80.$$
$$\therefore \quad PQ = 4\sqrt{5}.$$

If in the expression for PQ found in § 9, we substitute -3 for x_1, 2 for y_1, 5 for x_2 and -2 for y_2, we obtain

$$PQ = \sqrt{(-3-5)^2 + (2+2)^2}$$
$$= 4\sqrt{5},$$

the same result.

12. The particular cases in § § 10 and 11 illustrate what is known as the **continuity** of the formulæ in analytical geometry. Here continuity means, that *general* results which are obtained when the coordinates in the diagram used are all positive hold true in the same form for all points.

13. **To find the coordinates of the middle point of the distance between two given points P (x_1, y_1) and Q (x_2, y_2).**

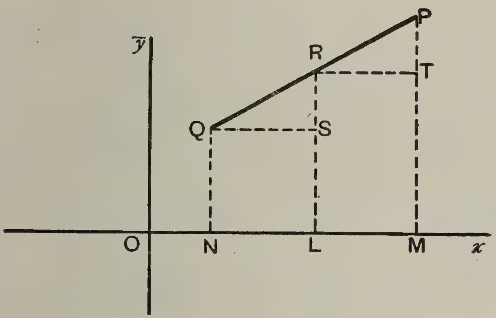

FIG. 9.

Let R (x, y) be the middle point of PQ.

Draw PM, QN, RL $\perp$ Ox; QS $\perp$ RL; RT $\perp$ PM.

From the equality of $\triangle$s PRT, RQS,

$$QS = RT \text{ and } RS = PT.$$

$$\therefore NL = LM,$$

$$\therefore \quad x - x_2 = x_1 - x.$$

$$\therefore \quad x = \frac{x_1 + x_2}{2}.$$

$$\because RS = PT,$$

$$\therefore \quad y - y_2 = y_1 - y.$$

$$\therefore \quad y = \frac{y_1 + y_2}{2}.$$

Thus the coordinates of R are

$$\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}.$$

14. To find the coordinates of the point dividing the distance between P $(x_1\ y_1)$ and Q $(x_2\ y_2)$ in the ratio of *m* to *n*.

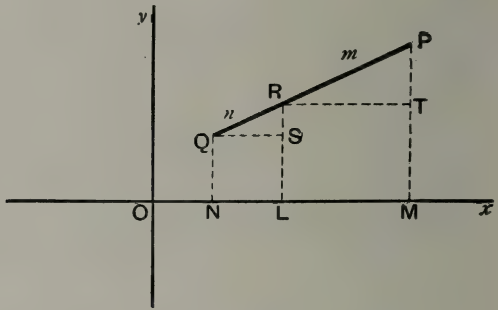

FIG. 10.

Let R (x, y) be the point dividing PQ such that $\frac{\text{PR}}{\text{RQ}} = \frac{m}{n}$.

Draw PM, QN, RL $\perp$ Ox; QS $\perp$ RL; RT $\perp$ PM.

From the similar $\triangle$s PRT, RQS

$$\frac{\text{RT}}{\text{QS}} = \frac{\text{PT}}{\text{RS}} = \frac{\text{PR}}{\text{RQ}} = \frac{m}{n}.$$

$$\because \quad \frac{\text{RT}}{\text{QS}} = \frac{m}{n},$$

$$\therefore \quad \frac{x_1 - x}{x - x_2} = \frac{m}{n}.$$

$$\therefore \quad mx - mx_2 = nx_1 - nx.$$

$$\therefore \qquad x = \frac{nx_1 + mx_2}{m + n}.$$

$$\because \qquad \frac{\mathsf{PT}}{\mathsf{RS}} = \frac{m}{n},$$

$$\therefore \qquad \frac{y_1 - y}{y - y_2} = \frac{m}{n}.$$

$$\therefore \qquad my - my_2 = ny_1 - ny.$$

$$\therefore \qquad y = \frac{ny_1 + my_2}{m + n}.$$

Thus the coordinates of R are

$$\frac{\mathbf{nx_1 + mx_2}}{\mathbf{m + n}}, \quad \frac{\mathbf{ny_1 + my_2}}{\mathbf{m + n}}.$$

15. If the point R be taken in PQ produced such that PR : RQ $= m : n$, and the coordinates of P, Q be (x_1, y_1), (x_2, y_2) it may be shown by a proof similar to that in the previous article that the coordinates of R are

$$\frac{\mathbf{mx_2 - nx_1}}{\mathbf{m - n}}, \quad \frac{\mathbf{my_2 - ny_1}}{\mathbf{m - n}}.$$

These results and also those of §§ 13 and 14 are the same for oblique and rectangular axes

16.—Exercises

1. Find the distance between the points (6, 5) and (1, – 7) and test your result by measurement on squared paper.

2. Find the distance between the points (2, – 3) and (– 1, 1) and test your result by measurement on squared paper.

3. Find the coordinates of the middle points of the st. lines joining the pairs of points in exercises 1 and 2 respectively and test the results by measurements on the diagrams.

4. Find, to two decimal places, the distance between $(-3, 7)$ and $(4, -4)$.

5. The vertices of a $\triangle$ are $(-2, 4)$, $(-8, -4)$ and $(7, 4)$. Find the lengths of its sides.

6. The vertices of a $\triangle$ are $(-1, 5)$, $(-4, -2)$, $(5, -3)$. Find (a) the lengths of the sides; (b) the lengths of the medians.

7. The vertices of a quadrilateral are $(4, 3)$, $(-5, 2)$, $(-3, -4)$, $(6, -2)$. Find the lengths of its sides, and also of its diagonals.

8. Find the coordinates of the middle point of the st. line joining $(3, -2)$ and $(-3, 2)$.

9. Find the points of trisection of the st. line joining $(1, 3)$ and $(6, 1)$.

10. The st. line joining P $(-4, -3)$ and Q $(6, -1)$ is divided at R (x, y) so that PR : RQ $= 5 : 2$. Show that $x = -2y$.

11. Find the length of the st. line joining the origin to $(a, -b)$.

12. The st. line joining the origin to P $(-4, 7)$ is divided at R, Q so that OR : RQ : QP $= 3 : 4 : 2$. Find the distance RQ.

13. The length of a st. line is 17 and the coordinates of one end are $(-5, -8)$. If the ordinate of the other end is 7, find its abscissa.

14. Find in its simplest form the equation which expresses the fact that (x, y) is equidistant from $(5, 2)$ and $(3, 7)$.

15. Find the centre and radius of the circle which passes through $(5, 2)$, $(3, 7)$ and $(-2, 4)$.

16. Find the points which are distant 15 from $(-2, -10)$ and 13 from $(2, 14)$.

17. Prove that the vertices of a rt.-∠d △ are equidistant from the middle point of the hypotenuse.

Suggestion:—Take the vertex of the rt. ∠ for origin and the sides which contain the rt. ∠ for axes.

18. In any △ ABC prove that

$$AB^2 + AC^2 = 2\,(AD^2 + DC^2),$$

where D is the middle point of BC.

Suggestion:—Take D *as origin,* DC *as axis of x and the* ⊥ *to* BC *at* D *as axis of y. Let* DC $= a$, *and the coordinates of* A *be* (x_1, y_1).

19. If D is a point in the base BC of a △ ABC such that BD : DC $= m : n$, show that

$$n\,AB^2 + m\,AC^2 = (m+n)\,AD^2 + n\,BD^2 + m\,DC^2.$$

Suggestion:—Take D *as origin,* DC *as axis of x and the* ⊥ *to* BC *at* D *as axis of y. Let* BD $= -ma$, DC $= na$, *and the coordinates of* A *be* (x_1, y_1).

20. The vertices of a △ are the points (x_1, y_1), (x_2, y_2), (x_3, y_3). Find the coordinates of its centroid.

21. The st. line joining A $(2, 1)$ to B $(5, 9)$ is produced to C so that AC : BC = 7 : 2. Find the coordinates of C.

22. The st. line joining A $(3, -2)$ to B $(-4, -6)$ is produced to C so that AC : BC = 3 : 2. Find the coordinates of C.

The Area of a Triangle

17. To find the area of the $\triangle$ of which the vertices are A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3).

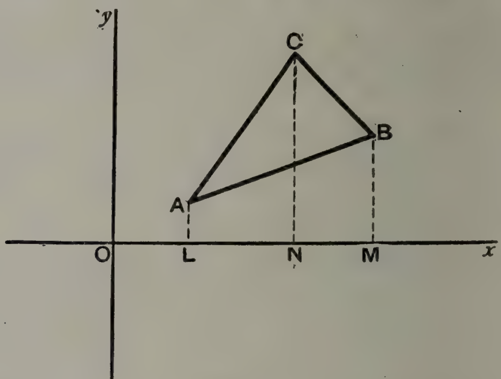

Fig. 11.

Draw the ordinates AL, BM, CN.

From the diagram,

$\triangle$ ABC = ALNC + CNMB − ALMB.

The area of a quadrilateral of which two sides are ∥ = half the sum of the ∥ sides × the distance between the ∥ sides.

$$\therefore \text{ALNC} = \tfrac{1}{2}(\text{AL} + \text{CN}) \times \text{LN} = \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1),$$

$$\text{CNMB} = \tfrac{1}{2}(\text{CN} + \text{BM}) \times \text{NM} = \tfrac{1}{2}(y_3 + y_2)(x_2 - x_3),$$

$$\text{ALMB} = \tfrac{1}{2}(\text{AL} + \text{BM}) \times \text{LM} = \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1).$$

$$\therefore \triangle \text{ABC} = \tfrac{1}{2}\left\{(y_1 + y_3)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) - (y_1 + y_2)(x_2 - x_1)\right\}.$$

Simplifying,

$$\triangle \text{ABC} = \tfrac{1}{2}\left\{\mathbf{x}_1(\mathbf{y}_2 - \mathbf{y}_3) + \mathbf{x}_2(\mathbf{y}_3 - \mathbf{y}_1) + \mathbf{x}_3(\mathbf{y}_1 - \mathbf{y}_2)\right\}.$$

NOTE.—*The points have been taken in circular order about the △ in the opposite direction to that in which the hands of a clock rotate; if they are taken in the same direction as the hands rotate, the formula will give the same result only it will appear to be negative; but, of course, the area of a △ must be positive.*

18. To find the area of the △ of which the vertices are (3, 2), (– 4, 3), (– 2, –4).

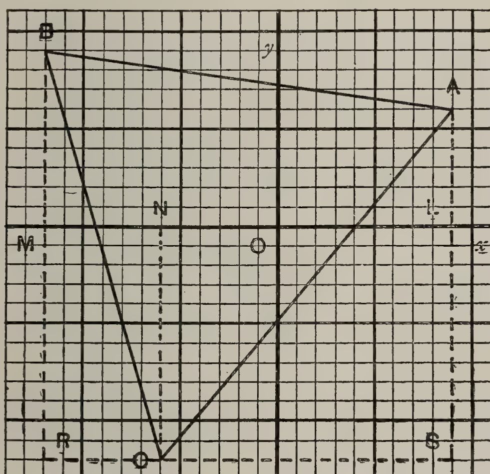

FIG. 12. (Unit = $\frac{3}{10}$ inch.)

Draw the diagram on squared paper. Draw the ordinates AL, BM, CN. Through C draw RCS ∥ Ox to meet AL, BM produced at S, R.

$$\triangle ABC = BRSA - \triangle BRC - \triangle ACS.$$

$$BRSA = \tfrac{1}{2}(BR + AS)\,RS = \tfrac{1}{2}(7+6)\times 7 = \frac{91}{2}.$$

$$\triangle \mathsf{BRC} = \tfrac{1}{2}\,\mathsf{BR} \times \mathsf{RC} = \tfrac{1}{2} \times 7 \times 2 = \frac{14}{2}.$$

$$\triangle \mathsf{ASC} = \tfrac{1}{2}\,\mathsf{AS} \times \mathsf{SC} = \tfrac{1}{2} \times 6 \times 5 = \frac{30}{2}.$$

$$\therefore \triangle \mathsf{ABC} = \frac{91 - 14 - 30}{2} = \frac{47}{2}.$$

If we substitute the coordinates of A, B and C in the formula of § 17, we obtain

$$\triangle \mathsf{ABC} = \tfrac{1}{2}\left\{3(3+4) + (-4)(-4-2) + (-2)(2-3)\right\}$$

$$= \tfrac{1}{2}(21 + 24 + 2) = \frac{47}{2};$$

the same result as before.

This illustrates the continuity of the symmetrical result found in § 17 for the area of a △.

19.—Exercises

1. Find, from a diagram, the area of the △ of which the vertices are (o, o), (a, b), (c, d). Check your result by using the formula of § 17.

2. Draw the following △s on squared paper and find their areas; checking your results by using the formula of § 17:—

(*a*) $(1, 4)$, $(-2, 2)$, $(5, -1)$;

(*b*) $(4, -2)$, $(-5, -1)$, $(-2, -6)$;

(*c*) $(0, 0)$, $(3, 4{\cdot}5)$, $(-2{\cdot}5, 4)$.

3. Find the area of the quadrilateral of which the vertices are $(3, 6)$, $(-2, 4)$, $(2, -2)$ and $(7, 3)$.

4. Find the area of the quadrilateral of which the vertices are $(0, 0)$, $(4, 0)$, $(3, 6)$ and $(-3, 3)$.

5. **D, E, F** are respectively the middle points of the sides **BC, CA, AB** of a △. Prove by the formula of § 17, taking **B** as origin and **BC** as axis of x, that △ **ABC** = 4 △ **DEF.**

6. Find the area of the △ of which the vertices are (x, y), $(3, 5)$, $(-2, 4)$; and thence show that if these points are in a st. line $5y - x = 22$.

7. Find the area of the △ **A** $(-3, 2)$, **B** $(7, 2)$, **C** $(3, 10)$; and show that the ⊥ from **A** to **BC** = **BC.**

8. A man starts from **O** and goes to **A**, from **A** to **B**, **B** to **C**, **C** to **D**, **D** to **O**. If **O** be taken as the origin and the coordinates of **A, B, C, D** are $(0, -3)$, $(8, 3)$, $(-4, 8)$, $(-4, 3)$, find the distance he has travelled, the unit being one mile.

9. Show from the formula for the area of a △ that **A** $(3, -2)$, **B** $(19, 10)$ and **C** $(7, 1)$ are in the same st. line. Find the ratio of **AC** to **CB.**

10. Show that if the coordinates of the vertices taken in order of a quadrilateral are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), its area is

$$\tfrac{1}{2}\{x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)\}.$$

11. In the △ **OAB**, **P** is taken in **OA**, **Q** in **AB** and **R** in **BO** so that **OP : PA = AQ : QB = BR : RO** = 3 : 1. Show that △ **PQR** : △ **OAB** = 7 : 16.

Loci

20. The definition of a locus (see Ontario H. S. Geometry, page 77) is:—

When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the locus of these points.

The condition which the points satisfy may be expressed in the form of an equation involving the coordinates of the points. For example, take the locus of the points of which the ordinate is equal to 3. This condition, which is expressed by the equation $y = 3$ [*i.e.*:—$0x + y = 3$], is satisfied by an infinite number of points, as (0, 3), (1, 3), (2, 3), (7, 3), (−4, 3), etc. All such points are on a st. line AB ∥ to Ox and 3

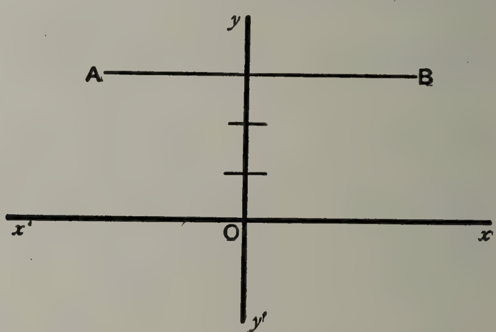

Fig. 13.

units above it; and this st. line contains no points which do not satisfy the condition. Thus the equation $y = 3$ represents the line AB.

Similarly the equation $y = -3$ represents a st. line $\parallel$ Ox and three units below it; $x = 3$ represents a st. line $\parallel$ Oy and three units to the right of the origin, and $x = -5$ a st. line $\parallel$ Oy and 5 units to the left of the origin.

For another example let us take the condition to be that the abscissa and ordinate of each point are equal. The points (0, 0), (1, 1), (2, 2), (4, 4), $(-1, -1)$, $(-5, -5)$, etc., satisfy this condition. It is expressed by the equation $y = x$. If we draw a diagram on

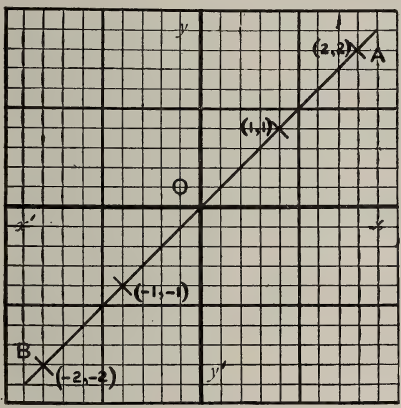

FIG. 14. (Unit = $\frac{4}{10}$ inch.)

squared paper, mark some of these points on it and join them we get a st. line AB bisecting the $\angle$s xOy and x'Oy' every point on which satisfies the given condition. Between O and (1, 1) there are an infinite number of points, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{10}, \frac{1}{10})$, $(\frac{9}{10}, \frac{9}{10})$, etc., which satisfy the condition, and so on continuously throughout the line. Thus the equation $y = x$ represents the line AB.

Again, we may consider the point which moves so that its distance from the origin is always 5. Its locus is plainly the circumference of a circle. Particular

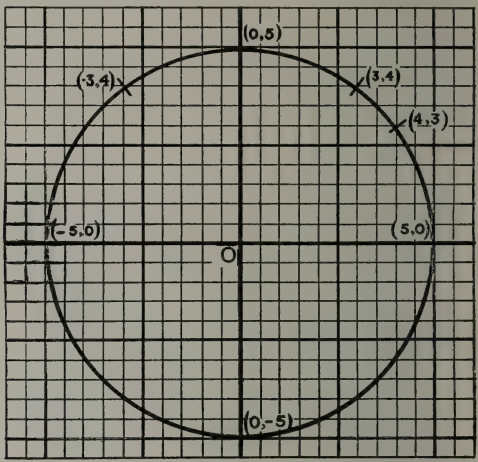

FIG. 15. (Unit = $\frac{2}{10}$ inch.)

points on this locus are (5, 0), (4, 3), (3, 4), (0, 5), $(-3, 4)$, etc., and its equation is $\sqrt{x^2 + y^2} = 5$, or $x^2 + y^2 = 25$.

21. In the equation of a locus the numbers that are the same for all points on the locus are called **constants**; while those that change in value continuously from point to point are called **variables**.

Thus, in the equation $x^2 + y^2 = 25$, x and y are variables and 25 is a constant.

22.—Exercises

1. Find four or five points on the locus represented by each of the following equations; and draw the locus on squared paper in each case:—

(*a*) $x = -4$; (*b*) $x + y = 0$; (*c*) $x - 2y = 0$;
(*d*) $3x + y = 0$; (*e*) $x = y + 4$; (*f*) $x^2 + y^2 = 169$.

2. A point moves so that its distance from the axis of x is 5 times its distance from the axis of y. Find the equation of its locus.

3. What locus is represented by the equation (*a*) $y = 0$; (*b*) $x = 0$?

4. A point moves so that it is equidistant from the origin and from (8, 0). Find the equation of its locus.

5. A point moves so that it is equidistant from the origin and from (3, – 5). Find the equation of its locus, and draw the locus on squared paper.

6. A point is equidistant from (1, – 2) and (– 3, – 4). Find the equation and draw the locus on squared paper.

7. A point moves so that its distance from (4, 3) is always 5. Find the equation and show that the locus passes through the origin.

8. The coordinates of the ends of the base of a △ are (– 2, – 3) and (4, – 1), and the length of the median drawn to the base is 6. Find the equation of the locus of its vertex.

9. The coordinates of the ends of the base of a △ are (0, 0) and (5, 0), and its area is 10. Show that the equation of the locus of its vertex is $y = 4$.

10. The coordinates of the ends of the base of a △ are (– 1, – 2) and (5, 1) and its area is 9. Find the equation of the locus of its vertex.

23. An equation connecting two variables x and y has an infinite number of solutions. For example, in the equation $y = 3x + 7$, if any value is given to x, the corresponding value of y may then be determined. Thus, when

$$\begin{array}{lrr}
(a)\ x = & 0,\ y = & 7, \\
(b)\ x = & 1,\ y = & 10, \\
(c)\ x = & 2,\ y = & 13, \\
(d)\ x = & -1,\ y = & 4, \\
(e)\ x = & -3,\ y = & -2, \\
(f)\ x = & \frac{1}{3},\ y = & 8,
\end{array}$$

etc.

The, in general, continuous line which passes through all the points (a), (b), (c), etc., is the locus represented by this equation.

Another equation as $4x + 3y = 8$ has also an infinite number of solutions, and if these two equations are solved together, the common solution obtained, in this case $x = -1$, $y = 4$, gives the coordinates of the point of intersection of the loci represented by the equations.

Sets of solutions which satisfy the equation $4x + 3y = 8$ are given in the following table:—

x	y
(d) -1	4
(g) 2	0
(h) 5	-4

If we plot these two sets of results on squared

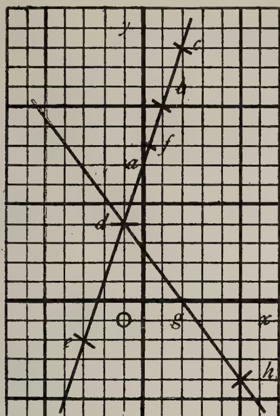

FIG. 16. (Unit = $\frac{1}{10}$ inch.)

paper, we see that the loci appear to be st. lines which intersect at the point (d) $(-1, 4)$.

24.—Exercises

1. Plot the following loci on squared paper and find the coordinates of their points of intersection:—

(a) $4x - y = 1$ and $x - 2y = -12$;
(b) $x + 2y = 7$ and $5x - 2y = 11$;
(c) $3x + 8y = -18$ and $4x + 3y = -1$;
(d) $3x + 4y = 0$ and $x^2 + y^2 = 100$;
(e) $3x - 5y + 45 = 0$ and $x^2 + y^2 = 169$.

2. Find the points where the locus $3x - 5y + 45 = 0$ cuts the axes.

3. Find the points where the locus $x^2 + y^2 = 6\,x$ cuts the axis of x.

4. Find the locus of a point such that the square of its distance from $(-a, o)$ is greater than the square of its distance from (a, o) by $2a^2$.

5. Find the equation of the locus of a point such that the square of its distance from $(-2, -1)$ is greater than the square of its distance from $(5, 3)$ by 11.

6. A $(1, 0)$ and B $(9, 0)$ are two fixed points and P is a variable point such that PB = 3 PA. Find the equation of the locus of P.

7. Plot the following loci and show that they are concurrent:—

$$3x + 4y = 10, \; 5x - 2y = 8, \; 4x + y = 9.$$

CHAPTER II

THE STRAIGHT LINE

25. To find the equation of a st. line in terms of the intercepts that it makes on the axes.

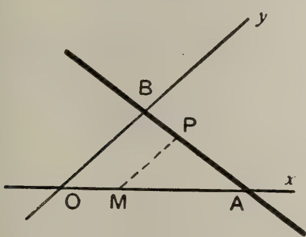

FIG. 17.

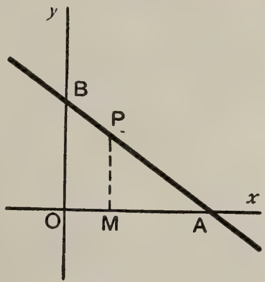

FIG. 18.

Let the st. line cut the axes at A, B so that OA $= a$, OB $= b$.

Take P (x, y) any point on the line, and draw PM $\parallel$ Oy and terminated in Ox at M.

From the similar $\triangle$s APM, ABO,

$$\frac{PM}{BO} = \frac{AM}{AO}.$$

$$\therefore \quad \frac{y}{b} = \frac{a-x}{a}.$$

$$\text{or,} \quad \frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} = 1.$$

NOTE.—*It is seen from the diagrams that both the proof and the form of the equation are the same for oblique and rectangular axes.*

26. To find the equation of the st. line passing through A (x_1, y_1) and B (x_2, y_2).

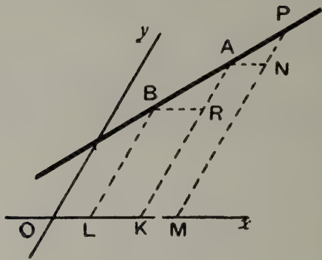

FIG. 19.

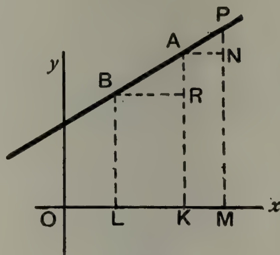

FIG. 20.

Take any point P (x, y) on the st. line.

Draw AK, BL PM ∥ Oy and terminated in Ox at K, L, M; and AN, BR ∥ Ox and respectively terminated in PM at N and AK at R.

From the similar △s PNA, ARB,

$$\frac{\text{AN}}{\text{BR}} = \frac{\text{PN}}{\text{AR}}.$$

$$\text{AN} = \text{KM} = \text{OM} - \text{OK} = x - x_1,$$

$$\text{BR} = \text{LK} = \text{OK} - \text{OL} = x_1 - x_2,$$

$$\text{PN} = \text{PM} - \text{NM} = \text{PM} - \text{AK} = y - y_1,$$

$$\text{AR} = \text{AK} - \text{RK} = \text{AK} - \text{BL} = y_1 - y_2$$

$$\therefore \frac{\mathbf{x - x_1}}{\mathbf{x_1 - x_2}} = \frac{\mathbf{y - y_1}}{\mathbf{y_1 - y_2}}.$$

NOTE.—*It is seen from the diagrams that both the proof and the form of the equation are the same for oblique and rectangular axes.*

27.—Exercises

1. The equation of the st. line passing through (4, 3) and (–2, 7) is by the formula of § 26

$$\frac{x-4}{4+2}=\frac{y-3}{3-7},$$

$$\text{or,}\quad 2x+3y=17.$$

To find the intercepts which this line makes on the axes, let $y=0$ and $\therefore\ x=8\frac{1}{2}$, let $x=0$ and $\therefore\ y=5\frac{2}{3}$. By § 25 the equation of the line may now be written

$$\frac{x}{8\frac{1}{2}}+\frac{y}{5\frac{2}{3}}=1.$$

This is clearly the same as $2x+3y=17$.

2. Write down the equations of the st. lines which make the following intercepts on Ox, Oy respectively:—

(*a*) 5, 2; (*b*) –4, –6; (*c*) 3, –8.

3. Find the equations of the st. lines through the following pairs of points:—

(*a*) (6, 2), (3, 1); (*b*) (–1, 2), (–3, –7); (*c*) (4, –6), (–7, 2). Find the intercepts these st. lines make on the axes.

4. Find the point where the st. line which makes intercepts –3 and 5 on Ox and Oy respectively is cut by the st. line $x=-5$.

5. Find the point where the st. line making intercepts 7 and 2 on Ox and Oy respectively meets the st. line through (–2, 7) and (5, –3).

6. Find the point where the st. line through (3, 5) and (–7, –1) meets the st. line through (–8, 2) and (6, 5).

7. Prove that (11, 4) lies on the st. line joining (3, -2) and (19, 10) and find the ratio of the segments into which the first point divides the join of the other two.

8. Find the equations of the sides of the $\triangle$ of which the vertices are (4, -2), (-5, -1), and (-2, -6). Find also the equations of the medians of the $\triangle$ and the coordinates of its centroid.

9. The vertices of a quadrilateral are (3, 6), (-2, 4), (2, -2) and (7, 3). Find the equations of the four sides. Find also the equations of the three diagonals of the complete quadrilateral, and show that the middle points of the diagonals are collinear. Find the equation of the st. line passing through the middle points of the diagonals.

10. Find the vertices of the $\triangle$ the sides of which are $11x - 3y = -45$, $5x - 11y = 47$ and $3x + 4y = 7$.

11. P (x_1, y_1) is any point and $\frac{x}{a} + \frac{y}{b} = 1$ cuts Ox, Oy at A, B respectively. Show that the area of the $\triangle$ PAB =

$$\tfrac{1}{2}(bx_1 + ay_1 - ab).$$

28. As explained in elementary algebra, the **degree of a term,** with respect to certain letters, is the number of such letters that occur as factors in the term.

$3x$, $-5y$, ax, by are terms of the first degree with respect to x and y.

$5x^2$, $3y^2$, $-2xy$, ax^2 are terms of the second degree with respect to x and y.

29. **Degree of an equation.** An equation is said to be of the first degree in x and y when it contains a term, or terms, of the first degree in x and y, but no term of a higher degree than the first.

The **general equation of the first degree** in x and y is

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0.$$

An equation is said to be of the second degree in x and y when it contains a term, or terms, of the second degree in x and y, but no term of a higher degree than the second.

The **general equation of the second degree** in x and y is

$$\mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0,$$

or, in a more convenient form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

30. To prove that an equation of the first degree always represents a st. line.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be *any* three sets of simultaneous values of x and y which satisfy the equation $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$.

Then, (1) $\mathrm{A}x_1 + \mathrm{B}y_1 + \mathrm{C} = 0.$

(2) $\mathrm{A}x_2 + \mathrm{B}y_2 + \mathrm{C} = 0.$

(3) $\mathrm{A}x_3 + \mathrm{B}y_3 + \mathrm{C} = 0.$

From (1) and (2),

$$\frac{\mathrm{A}}{y_1 - y_2} = \frac{\mathrm{B}}{x_2 - x_1} = \frac{\mathrm{C}}{x_1 y_2 - x_2 y_1}.$$

Dividing the three terms of (3) respectively by these equal fractions and 0 by any one of them,

$$x_3 (y_1 - y_2) + y_3 (x_2 - x_1) + x_1 y_2 - x_2 y_1 = 0.$$

Rearranging the terms, we get

(4) $x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0.$

From § 17 the area of the $\triangle$ formed by joining (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\tfrac{1}{2} \{x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)\}$$

and $\therefore$, from (4), in this case, the area of the $\triangle$ is zero.

This can only be so when the three points are in a st. line, and $\therefore$ as any three points the coordinates of which satisfy

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$$

are in a st. line, this equation must always represent a st. line.

31. The equation $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ may be changed to the form

$$\frac{x}{-\frac{\mathsf{C}}{\mathsf{A}}} + \frac{y}{-\frac{\mathsf{C}}{\mathsf{B}}} = 1,$$

and by comparing this with the equation of § 25,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

we see that the intercepts which the st. line $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ makes on the axes of x and y are respectively

$$-\frac{\mathsf{C}}{\mathsf{A}} \text{ and } -\frac{\mathsf{C}}{\mathsf{B}}.$$

The same results are obtained by alternately letting $y = 0$ and $x = 0$ in $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$.

32. To obtain the result of § 26 from the general equation of the first degree.

Let (x_1, y_1), (x_2, y_2) be fixed points on the st. line represented by the general equation, and we have

$$(1)\quad \mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0,$$
$$(2)\quad \mathsf{A}x_1 + \mathsf{B}y_1 + \mathsf{C} = 0,$$
$$(3)\quad \mathsf{A}x_2 + \mathsf{B}y_2 + \mathsf{C} = 0.$$

From (2) and (3),

$$(4)\quad \frac{\mathsf{A}}{y_1 - y_2} = \frac{\mathsf{B}}{x_2 - x_1} = \frac{\mathsf{C}}{x_1y_2 - x_2y_1}.$$

$\therefore$, from (1) and (4),

$$x(y_1 - y_2) + y(x_2 - x_1) + x_1y_2 - x_2y_1 = 0.$$

This equation is seen to be the same as

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

when the latter is cleared of fractions and simplified.

33. To find the equation of a st. line in terms of its inclination to the axis of x and its intercept on the axis of y.

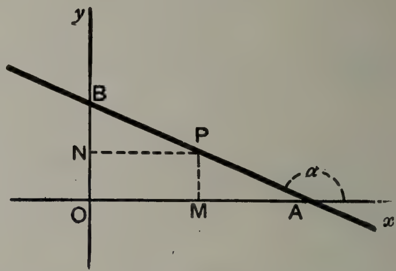

FIG. 21.

Let the st. line cut Ox, Oy at A, B respectively; $\angle$ BAx = a, and OB = b.

Take any point P (x, y) in the line, and draw PM $\perp$ Ox, PN $\perp$ Oy.

$$Tan \text{ BPN} = \frac{\text{BN}}{\text{PN}} = \frac{\text{BO} - \text{PM}}{\text{OM}} = \frac{b - y}{x}.$$

But, tan BPN = tan PAM = $- \tan a$.

$$\therefore -\tan a = \frac{b - y}{x},$$

$$\text{and } \therefore y = x \tan a + b.$$

If we let $\tan a = m$, the equation becomes

$$\mathbf{y = mx + b.}$$

In this equation m is called the **slope** of the line, and the $\angle\ a$, or $tan^{-1}m$, is always measured by a rotation in the positive direction from the positive direction of Ox, *i.e.*, the $\angle$ is traced out by a radius vector starting from the position Ax and rotating about A in the positive direction to the position AB.

NOTE.—*For oblique axes the proof and result are different from those given above for rectangular axes.*

34. The equation $Ax + By + C = 0$ may be changed to

$$y = -\frac{A}{B}x - \frac{C}{B},$$

from which by comparison with

$$y = mx + b,$$

it is seen that the slope of the st. line $Ax + By + C = 0$ is $-\frac{A}{B}$, and its intercept on the axis of y is $-\frac{C}{B}$.

35. To find the equation of a st. line in terms of the $\perp$ on it from the origin and the $\angle$ made by a positive rotation from Ox to this $\perp$.

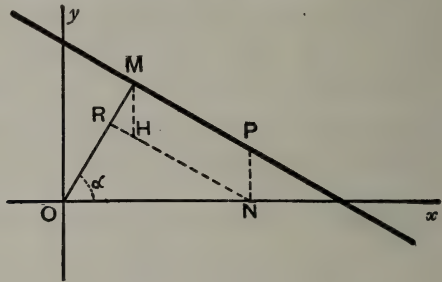

Fig. 22.

Let the $\perp$ OM from O to the line $= p$, and $\angle\ x\text{OM} = a$.

Take any point P (x, y) in the st. line.

Draw PN $\perp$ Ox, NR $\perp$ OM, MH $\parallel$ Oy to meet NR at H.

$$\text{OR} + \text{RM} = p.$$

$$\text{OR} = \text{ON}\ \cos \text{RON} = x\ \cos a.$$

$$\text{RM} = \text{MH}\ \cos \text{RMH} = \text{PN}\ \cos \text{MO}y = y\ \sin a.$$

$$\therefore\ \mathbf{x \cos a + y \sin a = p.}$$

NOTE.—*For oblique axes the proof and result are different from those given above for rectangular axes.*

36. To reduce the equation Ax + By + C = 0 to the form $x \cos a + y \sin a = p$, where p is always a positive quantity.

The equations

$$\begin{aligned} x \cos a + y \sin a - p &= 0 \\ \text{A}x + \text{B}y + \text{C} &= 0 \end{aligned}$$

will be identical if

$$\frac{\cos a}{\text{A}} = \frac{\sin a}{\text{B}} = \frac{-p}{\text{C}}.$$

If C is a positive quantity,

$$\frac{p}{\text{C}} = \frac{\cos a}{-\text{A}} = \frac{\sin a}{-\text{B}} = \frac{\sqrt{\cos^2 a + \sin^2 a}}{\sqrt{\text{A}^2 + \text{B}^2}} = \frac{1}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

$$\therefore \cos a = \frac{-\text{A}}{\sqrt{\text{A}^2 + \text{B}^2}},\ \sin a = \frac{-\text{B}}{\sqrt{\text{A}^2 + \text{B}^2}},\ \text{and } p = \frac{\text{C}}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

If C is a negative quantity, these results should be written

$$\cos a = \frac{\text{A}}{\sqrt{\text{A}^2 + \text{B}^2}},\ \sin a = \frac{\text{B}}{\sqrt{\text{A}^2 + \text{B}^2}},\ p = \frac{-\text{C}}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

Thus the equation is

$$\frac{\pm \text{A}x}{\sqrt{\text{A}^2 + \text{B}^2}} + \frac{\mp \text{B}y}{\sqrt{\text{A}^2 + \text{B}^2}} = \frac{\pm \text{C}}{\sqrt{\text{A}^2 + \text{B}^2}};$$

the upper signs being taken when C represents a positive quantity and the lower signs when C represents a negative quantity.

37. Ex. 1. Reduce the equation $3x + 4y - 12 = 0$ to the form $x \cos a + y \sin a = p$.

Here $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

Dividing the given equation by 5

$$\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}.$$

$\cos a = \frac{3}{5}$, $\sin a = \frac{4}{5}$, and $\therefore a = \tan^{-1}\left(\frac{4}{3}\right)$ while the $\perp$ from the origin on the line is $\frac{12}{5}$.

Ex. 2. Reduce the equation $x - y + 7 = 0$ to the form $x \cos a + y \sin a = p$.

Here $\sqrt{1^2 + 1^2} = \sqrt{2}$.

Dividing the given equation by $-\sqrt{2}$

$$-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$\text{or, } x \cos 135^\circ + y \sin 135^\circ = \frac{7}{\sqrt{2}}.$$

i.e., $\alpha = 135^\circ$, and the $\perp$ from the origin on the st. line is $\frac{7}{\sqrt{2}}$.

38. To find the equation of a st. line in terms of the coordinates of a fixed point on the line and the $\angle$ which the line makes with Ox.

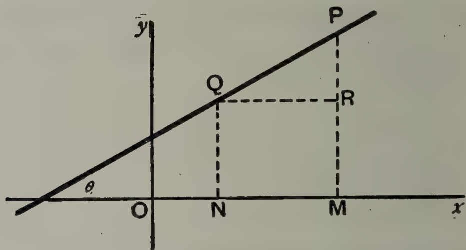

FIG. 23.

Let Q (x_1, y_1) be the fixed point and θ the $\angle$.

Take any point P (x, y) on the line and let QP $= r$. Draw PM, QN $\perp$ Ox and QR $\perp$ PM.

$$\text{QR} = \text{PQ} \cos \text{PQR}.$$

$$\text{QR} = \text{NM} = x - x_1, \text{ and } \angle \text{ PQR} = \angle\, \theta.$$

$$\therefore \quad \frac{x - x_1}{\cos \theta} = r.$$

PR = PQ *sin* PQR.

PR = PM − RM = PM − QN = $y - y_1$.

$$\therefore \quad \frac{y - y_1}{\sin\theta} = r.$$

$$\therefore \quad \frac{\mathbf{x} - \mathbf{x}_1}{\cos\theta} = \frac{\mathbf{y} - \mathbf{y}_1}{\sin\theta} = \mathbf{r}.$$

This form will frequently be found useful in problems that involve the distance between two points on a st. line.

If $\cos\theta = l$ and $\sin\theta = m$, the equation becomes:—

$$\frac{\mathbf{x} - \mathbf{x}_1}{\mathbf{l}} = \frac{\mathbf{y} - \mathbf{y}_1}{\mathbf{m}} = \mathbf{r}.$$

l and m are called the **direction cosines** of the st. line, and $l^2 + m^2 = 1$.

39. For convenience of reference the different forms of the equation of the st. line are here collected: —

(1) $Ax + By + C = 0$.

(2) $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(3) $\dfrac{x - x_1}{x_1 - x_2} = \dfrac{y - y_1}{y_1 - y_2}$.

(4) $y = mx + b$.

(5) $x \cos a + y \sin a = p$.

(6) $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = r$.

If the st. line passes through the origin (4) takes the convenient form:—

(7) $y = mx$.

If the st. line joins the origin to a fixed point (x_1, y_1), we get, by letting $x_2 = y_2 = 0$ in (3):—

$$(8)\quad \frac{x}{x_1} = \frac{y}{y_1}.$$

If the st. line passes through (x_1, y_1) and its slope is m, the equation is easily seen from (4) to be

$$(9)\quad y - y_1 = m(x - x_1).$$

40.—Exercises

1. Name the constants and variables in each of the nine equations of § 39. Explain the meaning of each constant. Which of these equations are of the same form for rectangular and oblique axes?

2. Draw the following st. lines on squared paper:—

(*a*) $x + 2y = 8$; (*b*) $3x - 7y = -5$; (*c*) $\frac{x}{3} + \frac{y}{5} = -1$; (*d*) $2x + 3y = 13$; (*e*) $3y = 4x$.

3. Find the equation of the st. line.

(*a*) through the origin and making an $\angle$ of 30° with Ox;

(*b*) through the origin and making an $\angle$ of 120° with Ox;

(*c*) through (0, 5) and making the $\angle$ $\tan^{-1} \frac{5}{7}$ with Ox;

(*d*) through (0, -3) and making the $\angle$ $\cos^{-1} \frac{3}{5}$ with Ox;

(*e*) through (-3, -4) and making the $\angle$ 45° with Ox.

4. In the $\triangle$ of which the vertices are (-2, 5), (3, -7), (4, 2)

(*a*) find the slope of each side;

(*b*) show that the medians are concurrent and find the centroid.

5. O (0, 0), A (6, 0), B (4, 6), C (2, 8) are the vertices of a quadrilateral. Show that the st. lines joining the

middle points of OA, BC, of AB, CO and of OB, AC are concurrent, and find the coordinates of their common point.

6. Find the equation to the st. line through $(-4, 3)$ that cuts off equal intercepts from the axes.

7. Find the length of the $\perp$ from the origin to the line $3x + 7y = 10$; find the $\angle$ which this $\perp$ makes with Ox.

8. What is the condition that the st. line $Ax + By + C = 0$ may

(*a*) pass through the origin;

(*b*) be $\parallel Ox$;

(*c*) be $\parallel Oy$;

(*d*) cut off equal intercepts from the axes;

(*e*) make $\angle\ 45°$ with Ox;

9. What must be the value of m if the line $y = mx + 7$ passes through $(-2, 5)$?

10. Find the values of m and b, if the st. line $y = mx + b$ passes through $(-2, 3)$ and $(7, 2)$.

11. Find the values of a and b, if the st. line $\frac{x}{a} + \frac{y}{b} = 1$ passes through $(-2, -5)$ and $(4, -2)$.

12. Show that the points $(4a, -3b)$, $(2a, 0)$, $(0, 3b)$ are in a st. line.

13. Show that the intercept made on the line $x = k$ by the lines $Ax + By + C = 0$ and $Ax + By + C' = 0$ is the same for all values of k.

14. $P\ (x_1, y_1)$ is any point and the lines $Ax + By + C = 0$ cuts Ox, Oy at N, R respectively. Show that $\triangle$ PNR =

$$\frac{C}{2\ AB}(Ax_1 + By_1 + C).$$

THE ANGLE BETWEEN TWO STRAIGHT LINES

41. To find the $\angle$ between two st. lines whose equations are given.

(i) Let the given equations be $y = m_1x + b_1$ and $y = m_2x + b_2$.

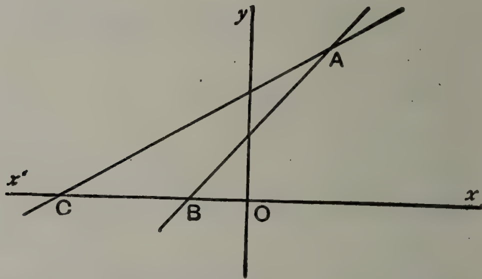

FIG. 24.

Let AB be the line $y = m_1x + b_1$ and AC be the line $y = m_2x + b_2$ when B, C are on the axis of x. Let $\angle$ BAC $= \theta$.

Then $m_1 = tan$ ABx, $m_2 = tan$ ACx.

$$\angle \theta = \angle \text{AB}x - \angle \text{AC}x.$$

$$\therefore \; tan\ \theta = \frac{tan\ \text{AB}x - tan\ \text{AC}x}{1 + tan\ \text{AB}x \,.\, tan\ \text{AC}x} = \frac{m_1 - m_2}{1 + m_1m_2}.$$

$$\therefore \qquad \theta = \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2}.$$

(ii) Let the given equations be $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$.

These equations may be changed to

$$y = -\frac{A}{B}x - \frac{C}{B} \text{ and } y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1}.$$

∴, writing $-\frac{A}{B}$ for m_1 and $-\frac{A_1}{B_1}$ for m_2 in the above result, the ∠ between the lines

$$= \tan^{-1}\frac{-\frac{A}{B}+\frac{A_1}{B_1}}{1+\frac{AA_1}{BB_1}} = \tan^{-1}\frac{A_1B - AB_1}{AA_1 + BB_1}.$$

42. **Condition of Parallelism.** If two st. lines are ∥, they make equal ∠s with the axis of x, *i.e.*, their slopes are the same.

∴, if their equations are $y = m_1x + b_1$ and $y = m_2x + b_2$, the condition is

$$\mathbf{m_1 = m_2}.$$

If their equations are $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$, the condition is

$$-\frac{A}{B} = -\frac{A_1}{B_1},$$

$$\text{or, } AB_1 - A_1B = 0.$$

This may also be written $\frac{A}{A_1} = \frac{B}{B_1}$; and we see that the equation $ax + by = k$ can be made to represent an infinite number of ∥ st. lines by giving different values to k; as:— $ax + by = k_1$, $ax + by = k_2$, etc.

43. **Condition of Perpendicularity.** If the st. lines $y = m_1x + b_1$, $y = m_2x + b_2$ are ⊥,

$$\tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2} = \frac{\pi}{2}.$$

$$\therefore \quad \frac{m_1 - m_2}{1 + m_1m_2} = \infty.$$

This will be true if $1 + m_1m_2 = 0$, and $\therefore$ the required condition is

$$m_1m_2 = -1.$$

Similarly, if $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$ are $\perp$.

$$AA_1 + BB_1 = 0.$$

The st. lines

$$Ax + By + C = 0$$
$$Bx - Ay + C_1 = 0$$

satisfy the above condition and $\therefore$ are $\perp$ to each other.

44.—Exercises

1. Find the $\angle$ between the st. lines

(*a*) $2x - 3y = 9$ and $x + 5y = 11$;

(*b*) $3x + 5y = 12$ and $(17\sqrt{3} + 30)x + 33y = 19$;

(*c*) $5y = 3x + 12$ and $5x + 3y = 17$;

(*d*) $4x + 7y = 13$ and $3x - y = 6$.

2. Find the equation of the st. line $\parallel$ to $6x - 7y = 13$ and passing through $(-2, -5)$.

Solution:—The required equation is

$$6(x + 2) - 7(y + 5) = 0;$$
i.e., $6x - 7y = 23$.

3. Find the equation of a st. line through $(-3, -5)$ and $\parallel$ to $9x + 4y = 18$.

4. Find the equation of the st. line drawn through $(-2, -5)$ and $\perp$ to $6x - 7y = 13$.

Solution:—The required equation is

$$7(x + 2) + 6(y + 5) = 0;$$
i.e., $7x + 6y + 44 = 0$.

5. Find the equation of the st. line drawn through $(4, 2)$ and $\perp$ $3x - 2y = 1$.

6. Find the equation of the st. line passing through $(-3, 5)$ and $\parallel$ to the st. line joining $(2, 6)$ and $(7, -1)$.

7. Find the equation of the st. line passing through $(2, 6)$ and $\perp$ the st. line joining $(-3, 5)$ and $(7, -1)$.

8. Find the equations of st. lines drawn through $(5, 7)$ which make $\angle$s $45°$ and $135°$ with Ox.

9. Find the equations of st. lines drawn through $(-5, -3)$ which make $\angle$s $30°$ and $150°$ with Ox.

10. Show that the $\perp$s from the vertices of the $\triangle$ $(1, -3)$, $(-5, -2)$, $(4, 7)$ to the opposite sides are concurrent; and find the coordinates of the orthocentre.

11. Show that the $\perp$s from the vertices of the $\triangle$ $(0, 0)$, $(a, 0)$, (b, c) to the opposite sides are concurrent; and find the orthocentre.

12. Find the ratio into which the $\perp$ from the origin on the st. line joining $(2, 6)$ and $(5, 1)$ divides the distance between these points.

13. Find the equations of the st. lines which pass through (h, k) and form with $y = mx + b$ an isosceles $\triangle$ of which the vertex is at the given point and each base $\angle = a$.

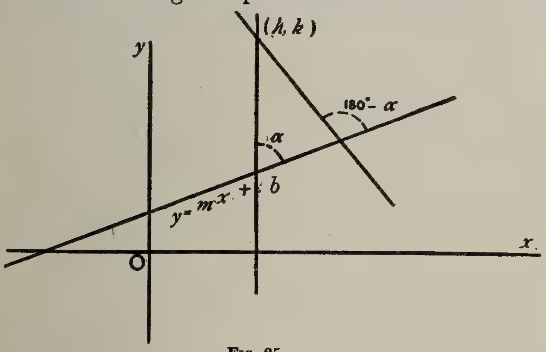

FIG. 25.

Solution:—Let $y - k = \mathsf{M}(x - h)$ represent one side of the $\triangle$ where the value of M is to be found.

$$Tan\ a = \frac{M - m}{1 + m M}.$$

$$\therefore \quad M = \frac{m + tan\ a}{1 - m\ tan\ a}.$$

$\therefore$ the equation of this side is

$$y - k = \frac{m + tan\ a}{1 - m\ tan\ a}(x - h).$$

If for a we substitute $108° - a$, the equation of the other side is found to be

$$y - k = \frac{m - tan\ a}{1 + m\ tan\ a}(x - h).$$

14. Find the equations of the st. lines passing through (2, 8) and making an $\angle$ of $30°$ with $3x - 12y = 7$.

15. Find the equations of the st. lines passing through $(-1, -2)$ and making an $\angle$ of $45°$ with $\frac{x}{7} + \frac{y}{5} = 1$.

16. Show that the equation of the st. line through (a, b) and making an $\angle$ of $60°$ with $x\ cos\ a + y\ sin\ a = p$ is $y - b = (x - a)\ tan\ (a \pm 30°)$.

17. Show that the right bisectors of the sides of the $\triangle$ (0, 0), $(a, 0)$, (b, c) are concurrent; and find their point of intersection.

18. Find the equation of a st. line $\perp$ to $Ax + By + C = 0$, and at a distance p from the origin.

PERPENDICULARS

45. To find the length of the $\perp$ from P (x_1, y_1) to $Ax + By + C = 0$.

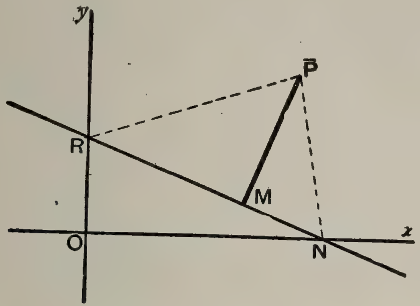

FIG. 26.

Draw PM $\perp$ the given st. line. Join P to N, R the points where the given st. line cuts Ox, Oy.

$$\triangle PRN = \tfrac{1}{2} PM \,.\, RN.$$

$$ON = -\frac{C}{A} \text{ and } OR = -\frac{C}{B},$$

$$\therefore RN = \sqrt{\frac{C^2}{B^2} + \frac{C^2}{A^2}} = \frac{C}{AB}\sqrt{A^2 + B^2}.$$

By the formula of § 17,

$$\triangle PRN = \tfrac{1}{2}\left\{x_1\left(-\frac{C}{B}\right) - \frac{C}{A}\left(y_1 + \frac{C}{B}\right)\right\}$$

$$= -\frac{C}{2AB}(Ax_1 + By_1 + C).$$

$$\therefore \tfrac{1}{2} PM \,.\, \frac{C}{AB}\sqrt{A^2 + B^2} = -\frac{C}{2AB}(Ax_1 + By_1 + C).$$

$$\therefore PM = -\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

The length of the $\perp$ is $\therefore$

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

In the diagram **AB** is the line represented by the equation $4x + 5y - 20 = 0$.

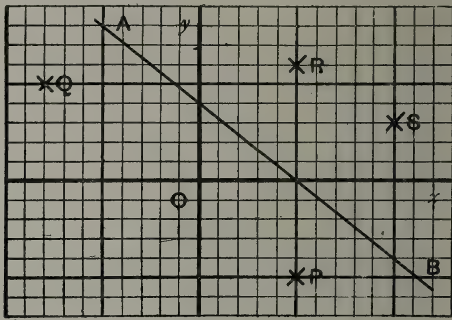

FIG. 27. (Unit = $\frac{1}{10}$ inch.)

If in the *expression* $4x + 5y - 20$ we substitute the coordinates of points **O, P, Q, R, S** which are not in the line, the following results are obtained.

For **O** $(0, 0)$, $4x + 5y - 20 = -20$.
" **P** $(5, -5)$, $4x + 5y - 20 = -25$.
" **Q** $(-8, 5)$, $4x + 5y - 20 = -27$.
" **R** $(5, 6)$, $4x + 5y - 20 = +30$.
" **S** $(10, 3)$, $4x + 5y - 20 = +35$.

In these results it will be observed that:—

For the origin the sign of the value of the expression is the same as the sign of the absolute term.

For other points that lie on the same side of the given st. line as the origin the signs of the values of the expression are the same as the sign of the result for the origin; while, for points on the side remote from the origin the signs of the values of the expression are different from the sign of the result for the origin.

A formal proof of these properties is given in the next article.

46. **To prove that the sign of the expression $Ax + By + C$ is different for points on opposite sides of the line $Ax + By + C = 0$.**

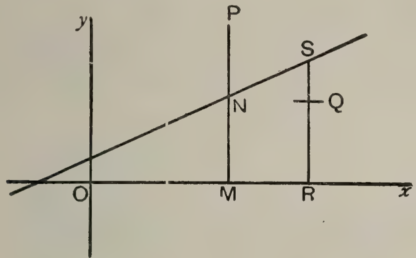

FIG. 28.

$P (x_1, y_1)$, $Q (x_2, y_2)$ are any points on opposite sides of $Ax + By + C = 0$.

Draw PM, $QR \perp Ox$ and let them cut the given line at N, S.

$$y_1 = PM = PN + NM;\ y_2 = QR = SR - SQ.$$

$$\therefore\quad Ax_1 + By_1 + C = Ax_1 + B.PN + B.NM + C,$$

and $$Ax_2 + By_2 + C = Ax_2 + B.SR - B.SQ + C.$$

But, $\because$ N and S are both on the given line,

$$Ax_1 + B.NM + C = 0,$$

and $$Ax_2 + B.SR + C = 0.$$

$$\therefore\quad Ax_1 + By_1 + C = B.PN,$$

and $$Ax_2 + By_2 + C = -B.SQ.$$

$\therefore$, since PN, SQ are both taken as positive quantities, $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite signs.

When $x = 0$ and $y = 0$ the expression $\mathrm{A}x + \mathrm{B}y + \mathrm{C}$ becomes C, $\therefore$ a point whose coordinates when substituted in $\mathrm{A}x + \mathrm{B}y + \mathrm{C}$ gives the same sign as C is on the same side of the st. line $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ as the origin.

47. **Sign of the Perpendicular.** It follows from the preceding article that, if the positive sign is always taken for $\sqrt{\mathrm{A}^2 + \mathrm{B}^2}$, when the sign of

$$\frac{\mathrm{A}x_1 + \mathrm{B}y_1 + \mathrm{C}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2}}$$

is the same as the sign of C, the point (x_1, y_1) and the origin are on the same side of the line $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$; and when the sign of this fraction is different from that of C, the point (x_1, y_1) and the origin are on opposite sides of $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$.

48. **To find the equations of the bisectors of the**

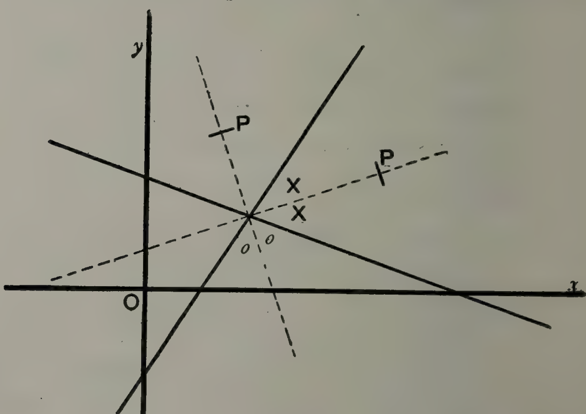

FIG. 29.

$\angle$s between the lines $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ and $\mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1 = 0$.

The $\perp$s to the st. lines from any point P (x, y) on either bisector are equal to each other,

$\therefore$ the required equations are

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A_1x + B_1y + C_1}{\sqrt{A_1{}^2 + B_1{}^2}}.$$

If the equations are so written that C and C_1 have the same sign and P is on the bisector of the $\angle$ that contains the origin, the $\perp$s from P have the same sign as the $\perp$s from the origin on the lines, and the equation of the bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \frac{A_1x + B_1y + C_1}{\sqrt{A_1{}^2 + B_1{}^2}}.$$

If P is on the bisector of the $\angle$ which does not contain the origin, the $\perp$s from P have opposite signs and the equation of the bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = -\frac{A_1x + B_1y + C_1}{\sqrt{A_1{}^2 + B_1{}^2}}.$$

49. To find the distance from (a, b) to $Ax+By+C=0$ in the direction whose direction cosines are l, m.

The equation of the st. line passing through (a, b) in the given direction is, by § 38,

$$\frac{x-a}{l} = \frac{y-b}{m} = r.$$

$$\therefore\ x = a + lr,\ y = b + mr.$$

Substituting these values for x and y in $Ax + By + C = 0$,

$$Aa + Alr + Bb + Bmr + C = 0.$$

$$\therefore\ r = -\frac{Aa + Bb + C}{Al + Bm}.$$

50. The length of the $\perp$ from (a, b) to $Ax + By + C = 0$ may be deduced from the result of § 49.

For, if $\dfrac{x-a}{l} = \dfrac{y-b}{m}$ and $Ax + By + C = 0$ are $\perp$ to each other,

$$\frac{A}{l} = \frac{B}{m} = \frac{Al + Bm}{l^2 + m^2} = Al + Bm,$$

since $l^2 + m^2 = 1$.

Also each of these fractions

$$= \frac{\sqrt{A^2 + B^2}}{\sqrt{l^2 + m^2}} = \sqrt{A^2 + B^2}.$$

$$\therefore\ Al + Bm = \sqrt{A^2 + B^2},$$

and the length of the $\perp$ is

$$\frac{Aa + Bb + C}{\sqrt{A^2 + B^2}}.$$

51. To find the equation of a line passing through the intersection of two loci.

The equation

$$k\,(\mathrm{A}x + \mathrm{B}y + \mathrm{C}) + l\,(\mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1) = 0, \qquad (1)$$

being of the first degree in x and y represents a st. line.

If (x_1, y_1) is the point of intersection of

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0 \qquad (2)$$

$$\text{and}\quad \mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1 = 0, \qquad (3)$$

the values x_1, y_1 substituted for x, y will plainly satisfy equation (1), and $\therefore$ the st. line (1) must pass through the point of intersection of the st. lines (2) and (3).

From the same reasoning the following more general theorem is seen to be true:—

If two equations are multiplied by any numbers and the results either added or subtracted, the resulting equation represents a locus that passes through the point (or points) of intersection of the loci represented by the first two.

52. *Example*—Find the equation of the st. line passing through the intersections of $17x - 7y = 9$, $3x + 19y = 34$ and $\perp$ to $11x - 4y = 13$.

$$17x - 7y - 9 + l\,(3x + 19y - 34) = 0$$

is a st. line passing through the intersection of the first two lines.

This equation may be written

$$(3l + 17)\,x + (19l - 7)\,y - 34l - 9 = 0.$$

If this line is $\perp$ to $11x - 4y = 13$,

$$11\,(3l + 17) - 4\,(19l - 7) = 0$$

$$\therefore\quad l = 5,$$

and the required equation is found to be

$$32x + 88y = 179.$$

53. To find the condition that the three st. lines

$$a_1 x + b_1 y + c_1 = 0 \ldots\ldots\ldots\ldots (1)$$

$$a_2 x + b_2 y + c_2 = 0 \ldots\ldots\ldots\ldots (2)$$

$$a_3 x + b_3 y + c_3 = 0 \ldots\ldots\ldots\ldots (3)$$

may be concurrent.

If the three st. lines are concurrent, the coordinates of the common point satisfy the three equations.

For that point, from (1) and (2),

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}.$$

Dividing the terms of (3) respectively by these equal fractions,

$$a_3 (b_1 c_2 - b_2 c_1) + b_3 (c_1 a_2 - c_2 a_1) + c_3 (a_1 b_2 - a_2 b_1) = 0.$$

This is the relationship that must hold among the constants in order that the lines may be concurrent.

54. Exercises

1. Find the length of the $\perp$

(*a*) from $(-4, 7)$ to $5x - 2y = 4$;

(*b*) from $(-4, -3)$ to $\frac{x}{2} + \frac{y}{3} = 1$;

(*c*) from $(3, -2)$ to $y = 7x + 1$;

(*d*) from $(-2, -7)$ to the st. line joining $(5, 3)$ and $(-3, -7)$;

(*e*) from the origin to the st. line joining $(7, 0)$ and $(0, -5)$.

2. Find the distance between the $\parallel$ lines $4x - 3y = 9$, $4x - 3y = 2$.

3. Find the distance between the $\parallel$ lines $ax + by + c_1 = 0$, $ax + by + c_2 = 0$.

4. Find the point in the line $\frac{x}{2} + \frac{y}{5} = -1$ such that its $\perp$ distance from the st. line joining $(2, 7)$, $(5, 3)$ is 8.

5. Find the equation of the st. line through the intersection of $3x - 2y = 12$, $5x + 4y = 9$ and $\parallel$ to $\frac{x}{3} + \frac{y}{4} = 1$. (See §§ 51 and 52).

6. Find the equation of the st. line joining the origin to the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

7. Find the distance from the point of intersection of $7x - 5y = 13$, $4x + 9y = 43$ to the line $12x = 5y$.

8. Find the equation of the st. line passing through the intersection of $y = mx + c$, $y = m_1x + c_1$ and $\perp$ to $\frac{x}{a} + \frac{y}{b} = 1$.

9. Show that the st. lines $x + 2y = 5$, $2x + 3y = 8$, $3x + y = 5$, $x + y = 3$ and $2x - y = 0$ are concurrent.

10. Find the condition that the lines $ax + hy + g = 0$, $hx + by + f = 0$, $gx + fy + c = 0$ are concurrent.

11. Find the equation of the st. line passing through the intersection of $y = mx + c$, $y = m_1x + c_1$ and also through (a, b).

12. Find the equation of the st. line joining the origin to the point of intersection of $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ and $\mathsf{A}_1x + \mathsf{B}_1y + \mathsf{C}_1 = 0$.

13. Find the distance from the orthocentre of the $\triangle$ **O** (0, 0), **A** (8, 0), **B** (3, 5) to the st. line **AB**.

14. Plot the lines $2x - 3y = 1$, $3x + y = 7$ on squared paper and find the intercepts that the bisectors of the $\angle$s between them make on the axis of y.

15. Find the equations of the bisectors of the $\angle$s between $5x - 12y = 17$ and $8x + 15y = 31$.

16. Show that the bisectors of the supplementary $\angle$s between $y = mx + a$ and $y = m_1x + a_1$ are $\perp$ to each other.

17. Find the equations of the bisectors of the $\angle$s between the st. lines joining (4, 5) and (−5, 2) respectively to (3, −7).

18. Show that the st. lines $x + 6y = 15$, $2x - 5y + 4 = 0$ and $9x + y = 29$ are concurrent.

19. The sides of a $\triangle$ are $3x + 4y = 15$, $12x - 5y = 17$, $24x + 7y = 30$. Plot the lines on squared paper and find the point where the bisectors of the interior $\angle$s of the $\triangle$ intersect.

20. Find the equation of the st. line passing through the intersection of the lines $2x - 3y = 6$ and $3x + 4y = 18$, and also through the middle point of the st. line joining $(-1, 2)$ and $(3, 4)$.

21. Find the distance from $(5, 3)$ in the direction in which the slope is $\frac{1}{\sqrt{3}}$ to the line $7x - 11y = 13$.

22. Find the distance from $(-4, 6)$ in the direction of which the slope is 1 to the line $\frac{x}{2} + \frac{y}{3} = 1$.

Draw the diagram on squared paper.

23. The sum of the distances from a point to the lines $x + 2y = 7$, $5x - 2y = 11$ is 7. Show that the locus of the point is a st. line which makes equal $\angle$s with the given st. lines.

24. Find the distance between the $\parallel$ lines $\frac{x}{a} + \frac{y}{b} = c$, $\frac{x}{a} + \frac{y}{b} = d$.

25. Find the equation of the st. line passing through **P** $(2, 5)$ and cutting Ox at **A**, Oy at **B** so that **AP** : **PB** $= 7:3$.

26. Find the equations of the st. lines passing through $(4, 7)$ and making an $\angle$ of $45°$ with $3x - 10y = 8$.

27. Find the equations of the st lines passing through $(-4, -7)$ and forming an equilateral $\triangle$ with $3x - 2y = 7$.

28. Find the equations of the st. lines drawn $\parallel$ to $5x - 12y = 9$ and at a distance 5 from it.

29 Find the equations of the two st. lines which pass through $(4, 7)$ and are equally distant from **A** $(7, 3)$, **B** $(3, -1)$. Find also the distances from **A** and **B** to these lines.

3). Find the point in $4x - 3y = 12$ which is equally distant from $(2, 7)$ and $(4, -1)$.

31. Having given the length of the base and the difference of the squares of the other two sides of a $\triangle$, prove that the locus of its vertex is a st. line $\perp$ the base.

32. Find the equations of the st. lines which are at a distance 2 from the origin and which pass through the intersection of $x - 7y + 11 = 0$ and $3x + 4y - 17 = 0$.

33. Find the equation of the st. line $\parallel$ to $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ and at a distance p from the origin.

34. Find the equation of the st. line passing through (h, k) and $\perp$ to $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$.

35. The equations of the sides of a $\triangle$ are $5x + 3y - 15 = 0$, $2x - y + 4 = 0$, $3x - 7y - 21 = 0$. (*a*) Show that the $\perp$s from the vertices to the opposite sides are concurrent and find the coordinates of the orthocentre. (*b*) Show that the right bisectors of the sides are concurrent and find the coordinates of the circumcentre. (*c*) Show that the centroid is at a point of trisection of the st. line joining the orthocentre to the circumcentre.

CHAPTER III

THE STRAIGHT LINE CONTINUED. TRANSFORMATION OF COORDINATES

55. An equation of the second degree may represent two st. lines.

For example, $2x^2 - 5xy + 3y^2 = 0$ is the same as $(x - y)(2x - 3y) = 0$, and will be true for all values of x and y which make either of the factors $x - y$ or $2x - 3y$ equal to zero, and $\therefore$ all points on the st. lines $x - y = 0$, $2x - 3y = 0$ are on the locus represented by $2x^2 - 5xy + 3y^2 = 0$.

Similarly, an equation of the third degree may represent three st. lines, one of the fourth degree may represent four st. lines, etc.

56. **The general equation $ax^2 + 2hxy + by^2 = 0$ represents two st. lines passing through the origin.**

Solving as a quadratic in x

$$x = \frac{-h \pm \sqrt{h^2 - ab}}{a} y,$$

from which it is seen that the given equation is equivalent to

$$\{ax + hy + y\sqrt{h^2 - ab}\}\{ax + hy - y\sqrt{h^2 - ab}\} = 0,$$

and $\therefore$ represents the two st. lines

$$ax + hy + y\sqrt{h^2 - ab} = 0$$
$$ax + hy - y\sqrt{h^2 - ab} = 0,$$

both of which pass through the origin.

If $h^2 > ab$, the lines are both real.

If $h^2 = ab$, the lines are coincident.

If $h^2 < ab$, the lines are imaginary, and we have two imaginary st. lines passing through the real point (0, 0).

57. **To find the $\angle$ between the two st. lines represented by $ax^2 + 2hxy + by^2 = 0$.**

The given equation may be written

$$y^2 + 2\frac{h}{b}xy + \frac{a}{b}x^2 = 0.$$

If $y - m_1x$ and $y - m_2x$ are the factors of the expression on the left hand side of this equation,

$$m_1 + m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}.$$

$$\therefore m_1^2 + 2m_1m_2 + m_2^2 = \frac{4h^2}{b^2}.$$

$$4m_1m_2 = \frac{4a}{b}.$$

$$\therefore (m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2},$$

$$\text{and } m_1 - m_2 = \frac{2\sqrt{h^2 - ab}}{b}.$$

If then θ is the $\angle$ between the st. lines, by § 41,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2\sqrt{h^2 - ab}}{b} \times \frac{b}{a + b}$$

$$= \frac{2\sqrt{h^2 - ab}}{a + b}.$$

$$\therefore \theta = \tan^{-1} \frac{2\sqrt{h^2 - ab}}{a + b}.$$

Condition of perpendicularity. If $\theta = 90°$,

$\tan \theta = \infty$. This will be the case if

$$a + b = 0.$$

58. **To find the equation of the st. lines which bisect the $\angle$s between the st. lines represented by $ax^2 + 2hxy + by^2 = 0$.**

Let the given equation represent the st. lines

$$y - m_1x = 0,\ y - m_2x = 0, \text{ so that}$$

$$m_1 + m_2 = -\frac{2h}{b},\quad m_1m_2 = \frac{a}{b}.$$

The equations of the bisectors of the $\angle$s between these lines are

$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} + \frac{y - m_2x}{\sqrt{1 + m_2^2}} = 0 \text{ and } \frac{y - m_1x}{\sqrt{1 + m_1^2}} - \frac{y - m_2x}{\sqrt{1 + m_2^2}} = 0.$$

These equations may be combined into

$$\frac{(y - m_1x)^2}{1 + m_1^2} - \frac{(y - m_2x)^2}{1 + m_2^2} = 0.$$

Simplifying and dividing by $m_2 - m_1$,

$$(m_1 + m_2)\, y^2 - 2\,(m_1m_2 - 1)\, xy - (m_1 + m_2)\, x^2 = 0.$$

Substituting and multiplying by b.

$$h\,(x^2 - y^2) - (a - b)\, xy = 0.$$

59. To find the relationship that must connect the constants in the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in order that this equation may represent two st. lines.

If the given equation represents two st. lines, it must be equivalent to two equations of the form $y - m_1x - b_1 = 0$, $y - m_2x - b_2 = 0$, from either of which y can be expressed in terms of the first degree of x.

Solving the given equation for y, we obtain

$$y = \frac{-(hx + f) \pm \sqrt{(hx + f)^2 - b(ax^2 + 2gx + c)}}{b}.$$

In order that these values of y may be in terms of the first degree of x, the expression under the radical sign must be a perfect square; *i.e.*,

$$(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc$$

is a perfect square for all values of x.

$$\therefore (hf - bg)^2 = (h^2 - ab)(f^2 - bc).$$

Simplifying, we get the condition in the form

$$2fgh - af^2 - bg^2 - ch^2 + abc = 0.$$

60.—Exercises

1. Show that the following equations represent two st. lines and find the separate equations of the lines :—

(*a*) $x^2 - (a + b)x = -ab$; (*b*) $x^2 - y^2 = 0$;

(*c*) $x^2 - 3xy = 0$; (*d*) $8x^2 + 3y^2 = 10xy$;

(*e*) $xy + bx = ay + ab$; (*f*) $3x^2 - 10xy + 3y^2 - 11x - 7y - 20 = 0$.

2. Show that $2x^2 - 7xy + 6y^2 + 2x - 5y - 4 = 0$ represents two st. lines and find the slope of each.

3. Interpret the locus represented by $xy = 0$.

4. Find the $\angle$s between the st. lines in 1. (d), (e) and (f).

5. Find the condition that $axy + bx + cy + d = 0$ may represent two st. lines.

6. Find the value of B for which the equation $3x^2 - 10xy + \mathsf{B}y^2 - 2x - 2y = 21$ will represent two st. lines.

7. Find the single equation which represents the two st. lines passing through (5, 3) and making an equilateral $\triangle$ with the axis of x.

8. Prove that $y^2 - 2xy \sec a + x^2 = 0$ represents two st. lines through the origin and inclined to each other at an $\angle = a$. Show also that one of these lines makes the same $\angle$ with the axis of x that the other makes with the axis of y

Transformation of Coordinates

61. It is often necessary to change the coordinates involved in a problem into a different set which are referred to axes drawn

(*a*) from a new origin, or

(*b*) in directions different from the original axes.

62. **To change from a pair of axes to another pair which are ∥ to the former, but have a different origin.**

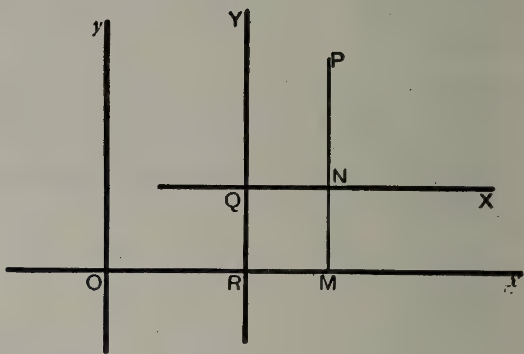

Fig. 30.

Q (h, k) is the new origin.

Let P (x, y) be any point referred to Ox and Oy and X, Y the coordinates of the same point referred to the new axes QX and QY.

Draw PNM ⊥ to Ox and QX, and let YQ cut Ox at R.

$$x = \text{OM} = \text{OR} + \text{QN} = h + \text{X}.$$

$$y = \text{PM} = \text{QR} + \text{PN} = k + \text{Y}.$$

Thus, if for x, y respectively we substitute $h + \mathsf{X}$, $k + \mathsf{Y}$ in any equation the origin is changed to the point (h, k).

To return to the original origin the substitutions would be $\mathsf{X} = x - h$, $\mathsf{Y} = y - k$.

63. **To change the direction of the axes, without changing the origin, the axes being rectangular.**

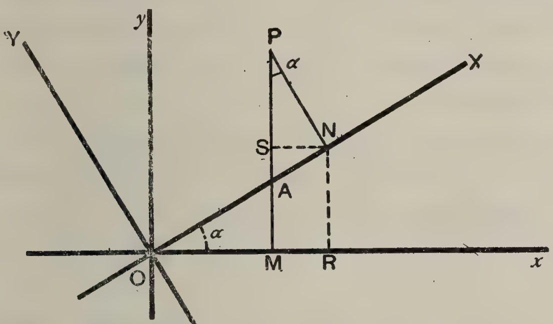

FIG. 31.

Let P (x, y) be any point referred to Ox, Oy; and X, Y the coordinates of the same point referred to axes OX, OY such that $\angle$ XOx = a.

Draw PM $\perp$ Ox, PN $\perp$ OX, NR $\perp$ Ox, NS $\perp$ PM.

$$\angle \text{NPS} = 90^\circ - \angle \text{NAP} = 90^\circ - \angle \text{MAO} = a.$$

$$x = \text{OM} = \text{OR} - \text{NS} = \text{X} \cos a - \text{Y} \sin a.$$

$$y = \text{PM} = \text{NR} + \text{PS} = \text{X} \sin a + \text{Y} \cos a.$$

Thus, if for x, y we substitute respectively $\text{X} \cos a - \text{Y} \sin a$, $\text{X} \sin a + \text{Y} \cos a$, the axes are rotated in the positive direction through an $\angle$ a.

64. By § 35, in an equation of the form $x \cos \alpha + y \sin \alpha = p$, the length of the $\perp$ from the origin on the st. line is the absolute term p.

If, without changing the direction of the axes, the origin be transferred to (x_1, y_1) the equation becomes

$$(x + x_1) \cos \alpha + (y + y_1) \sin \alpha = p,$$

$$i.e., \ x \cos \alpha + y \sin \alpha = p - x_1 \cos \alpha - y_1 \sin \alpha.$$

The new equation is of the same form as the old one except that the absolute term is now $p - x_1 \cos \alpha - y_1 \sin \alpha$.

This absolute term is then the length of the $\perp$ from the new origin to the st. line; or, reverting to the original origin, the length of the $\perp$ from (x_1, y_1) to the line $x \cos \alpha + y \sin \alpha = p$ is $p - x_1 \cos \alpha - y_1 \sin \alpha$.

This is the same as the result that would be obtained by using the formula of § 45.

65.—Exercises

1. What does the equation $2x^2 - 11xy + 12y^2 + 7x - 13y + 3 = 0$ become when the origin is changed to the point (1, 1) the directions of the axes being unchanged?

2. Transform the equation $x^2 + xy - 7x - 4y + 12 = 0$ to || axes through $(4, -1)$.

3. Find the point that must be taken as origin, the directions of the axes being unchanged, in order that the terms of the first degree in x and y may vanish from the equation $x^2 + y^2 + 5x - 9y + 17 = 0$. Find also what the equation becomes.

4. Show that the terms of the first degree in x and y will vanish from the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, if the origin be changed to $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$, the directions of the axes being unchanged.

5. Transform the equation $Ax + By + C = 0$ by rotating the axes through an $\angle$ of $30°$.

6. Find what the equation $x^2 - y^2 = a^2$ becomes when the axes are turned through an $\angle$ of $45°$, the origin remaining the same.

7. Show that the equation $x^2 + y^2 = a^2$ is not changed when the axes are turned through any $\angle\ \alpha$, the origin remaining the same.

8. Find what the equation $33x^2 - 34\sqrt{3}xy - y^2 = 0$ becomes when the axes are turned through an $\angle$ of $60°$, the origin remaining the same.

9. Find the smallest positive $\angle$ through which the axes must be turned in order that the coefficient of xy in the equation $59x^2 + 24\,xy + 66y^2 = 250$ may vanish; and also find what the equation becomes.

10. Show that the term involving xy in the expression $ax^2 + 2\,hxy + by^2$ will vanish, if the axes are turned through the $\angle$.

$$\tfrac{1}{2}\tan^{-1}\frac{2h}{a-b}.$$

66.—Review Exercises

1. Find the distances between the following pairs of points:—

(a) $(-2, 7), (6, -2)$;

(b) $(2a + b, a - 2b), (a - b, 3a + b)$;

(c) $(a \cos \alpha, a \cdot \sin \alpha), (-b \cos \alpha, -b \sin \alpha)$.

Verify the result in (b), on squared paper, when $a = 1$, $b = -2$.

2. A $(-5, -1)$, B $(4, 6)$ are two given points, P is taken in AB and Q in AB produced such that AP : PB = AQ : QB = 5 : 3. Find the coordinates of P and Q.

3. Find the area of the △ of which the vertices are $(3a, 2b)$, $(2a, 3b)$ and (a, b).

4. Find the area of the △ contained by the lines $2x + 11y + 43 = 0$, $9x + 8y - 14 = 0$ and $7x - 3y + 26 = 0$.

5. Find the ⊥ distance from $(-2, 3)$ to the line $\frac{2x}{3} - \frac{y}{2} = 6$.

Should the result be considered positive or negative and why?

6. Find the condition that the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) may lie in a st. line.

7. Find the locus of a point such that the square of its distance from $(-3, -7)$ exceeds the square of its distance from $(5, 0)$ by 43.

8. Prove that the equation $Ax + By + C = 0$ represents a st. line.

9. Find the equation of the st. line which is equidistant from the ∥ lines $ax + by = c$, $ax + by = d$.

10. Find the equation of a st. line which makes an $\angle\ \alpha$ with $\mathsf{O}y$ and cuts off an intercept b from $\mathsf{O}x$.

11. Show that the st. lines $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$, $\mathsf{A}_1x + \mathsf{B}_1y + \mathsf{C}_1 = 0$ are $\parallel$, if $\mathsf{AB}_1 = \mathsf{A}_1\mathsf{B}$.

12. Explain the meaning of the constants in the equations

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r.$$

13. Show that the line $y = x \tan \alpha$ passes through the point $(a \cos \alpha, a \sin \alpha)$, and find the equation of the $\perp$ to the line at that point.

14. Find the $\angle$ between the st. line joining $(-4, 5)$, $(5, 1)$ and the st. line joining $(3, 7)$, $(-6, -3)$.

15. Find the values of m and b such that the line $y = mx + b$ will pass through $(3, -2)$ and $(-1, -5)$.

16. Find the equation of the st. line which passes through $(2, -2)$, and makes an $\angle$ of $150°$ with $\mathsf{O}x$.

17. Find the length of the st. line drawn from (h, k), in the direction inclined at $\angle\ \alpha$ to $\mathsf{O}x$, and terminated in the line $y = mx + b$.

18. Find the equation of the st. line through (h, k), and $\parallel$ to $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

19. Show that the st. lines $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$, $\mathsf{A}_1x + \mathsf{B}_1y + \mathsf{C}_1 = 0$ are $\perp$ to each other, if $\mathsf{AA}_1 + \mathsf{BB}_1 = 0$.

20. Write the equation of the st. line which is $\perp$ $ax - by = c$, and cuts off an intercept $= d$ from $\mathsf{O}y$.

21. Find which of the following points are on the origin side of $\dfrac{x}{3} - \dfrac{y}{5} = 1$:— $(5, 3)$, $(-2, -8)$, $(2, -2)$, $(-6, -14)$, $(-7, -17)$. Illustrate by a diagram on squared paper.

22. Show that the points (2, 6), (1, 11), (– 4, 7), (– 3, 3) are in the four different angular spaces made by the lines

$$\frac{x}{3} + \frac{y}{5} = 1 \text{ and } \frac{x}{5} - \frac{y}{9} = -1.$$

Illustrate by a diagram on squared paper.

23. Find the values of a for which the lines $2x - ay + 1 = 0$, $ax - 6y - 1 = 0$, $18x - ay - 7 = 0$ are concurrent; and find also the coordinates of the respective points of intersection.

24. Show that the condition that the lines $ax + by = 1$, $cx + dy = 1$, $hx + ky = 1$ are concurrent is the same as the condition that the the points (a, b), (c, d), (h, k) are collinear.

25. Find the $\angle$ contained by the lines $4x - 7y + a = 0$, $3x + 11y + b = 0$.

26. Find the equation of the st. line passing through the intersection of $4x - 7y + a = 0$ and $3x + 11y + b = 0$ and making an $\angle$ of 45° with the axis of x.

27. Find the equation of the st. line passing through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$, $y = mx + c$ and also through $(d, 0)$.

28. Show that the equation of the st. line joining the intersection of $x \cos \alpha + y \sin \alpha = p$, $x \cos \beta + y \sin \beta = p$ to the origin is $y = x \tan \dfrac{\alpha + \beta}{2}$.

29. Find the length of the $\perp$ from (a, b) to $\frac{x}{a} + \frac{y}{b} = 1$.

30. Find the equation of the st. line through $(-5, 1)$ and $\parallel$ to $3x + 12y = 17$.

31. Find the equation of the st. line through $(8, -2)$ and $\perp$ to $7x = y + 4$.

32. Find the coordinates of the four points each of which is equally distant from the three lines $\frac{x}{4} - \frac{y}{3} = 1$, $\frac{x}{12} + \frac{y}{5} = 1$, $\frac{x}{24} - \frac{y}{7} = -1$.

33. Find the coordinates of the foot of the $\perp$ from $(3, 5)$ to the st. line joining $(-1, -2)$ and $(8, 1)$.

34. Find the separate equations of the st. lines represented by $3x^2 + 14xy - 2y^2 = 0$.

35. Find the product of the $\perp$s drawn from $(3, -2)$ to the st. lines represented by $5x^2 + 12xy + 2y^2 = 0$.

36. Show that the $\angle$ between the lines $y = mx + a$, $y = nx + b$ is $tan^{-1} \frac{m - n}{1 + mn}$.

37. Find the tangent of the $\angle$ between the st. lines represented by $5x^2 - 8xy - y^2 = 0$.

38. Find the equations of the st. lines which pass through $(3, 6)$, and are inclined at an $\angle$ of $45°$ to $\frac{x}{5} + \frac{y}{7} = 1$.

39. Show that the st. line joining the point $(1, 1)$ to the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ with $\frac{x}{b} + \frac{y}{a} = 1$ passes through the origin.

40. Show that the equation of the st. line passing through the intersection of $x \cos \alpha + y \sin \alpha = p$, $x \cos \beta + y \sin \beta = q$, and $\parallel$ to $x + y = k$ is

$$(x + y) \sin (\alpha - \beta) + p (\sin \beta - \cos \beta) + q (\cos \alpha - \sin \alpha) = 0.$$

41. Find the equation of the st. line passing through the intersection of $5x - 7y = 16$, $2x - 3y = 7$ and $\perp$ to $6x - 4y = 19$.

42. Find the equations of the st. lines drawn through the vertices and $\parallel$ to the opposite sides of the $\triangle$ of which the equations of the sides are $3x + 11y = 23$, $4x - 9y = 11$, $7x - 2y = -31$.

43. Show that the lines $5x + y = 4$, $2x + y = 2$, $3x + 3y = 4$ are concurrent; and find the coordinates of their common point.

44. Find the equation of the st. line passing through the intersection of $ax + by + c = 0$, $fx + gy + h = 0$, and (*a*) also through the origin; (*b*) $\perp$ to $x + y = k$.

45. Find the equations of the st. lines which bisect the $\angle$s between the lines $12x - 5y = 17$, $8x + 15y = 13$.

46. Find the equation of the st. line which passes through the point of intersection of the lines $5x + y = 4$, $4x - 9y = 11$, and is $\perp$ to the former.

47. Show that the points $(4, 2)$, $(6, 2)$, $(5, 2 + \sqrt{3})$ are the vertices of an equilateral $\triangle$.

48. Find the locus of a point which moves so that the sum of its distances from the axes is 10. Trace the locus on squared paper.

49. Find the locus of a point which moves so that the difference of its distances from the axes is 10. Trace the locus on squared paper.

50. Find the equations of the st. lines each of which passes through $(-5, -3)$ and is such that the part of it between the axes is divided at the given point in the ratio $7 : 3$.

51. Find the equation of the st. line which passes through $(3, -2)$, and is $\perp$ to $4x + y + 12 = 0$.

52. Find the equation of the right bisector of the st. line joining (a, b) and (h, k).

53. Two st. lines are drawn through $(0, -3)$ such that the $\perp$s on them from $(-6, -6)$ are each of length 3. Find the equation of the st. line joining the feet of the $\perp$s.

54. Find the $\angle$ of inclination of the lines $ax + by = c$, $(a + b)x - (a - b)y = d$.

55. A st. line is drawn through $(2, -4)$ and $\perp$ to $7x - 3y = 11$. Find the equations of the bisectors of the $\angle$s between the $\perp$ and the given st. line.

56. Find the equation of the st. lines which bisect the $\angle$s between the lines represented by $x^2 + 2xy \sec\theta + y^2 = 0$.

57. Find the value of h for which the equation $3x^2 + hxy - 10y^2 + x + 29y - 10 = 0$ will represent two st. lines.

58. Show that, if the axes are rotated through an $\angle$ of $45°$, the term containing xy vanishes from the equation $x^2 + 2xy \sec\theta + y^2 = 0$; and the separate equations of the two st. lines become $x = \pm y \tan\dfrac{\theta}{2}$.

59. Three vertices of a $\|$gm are $(3, 4)$, $(-3, 1)$, $(5, -2)$. Find the coordinates of the fourth vertex.

60. Prove that the two st. lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

61. What must be the value of m, if the line $y = mx - 5$ passes through the intersection of $7x - 11y = 14$ and $5x + 2y = -11$.

62. Find the area of the $\triangle$ contained by the lines $x + y = 12$, $2x - y = 12$, $x - 2y = -12$.

CHAPTER IV

THE CIRCLE

67. A **circle** is the locus of the points that lie at a fixed distance from a fixed point.

The fixed point is the **centre** and the fixed distance is the **radius** of the circle.

68. **To find the equation of a circle having its centre at the origin.**

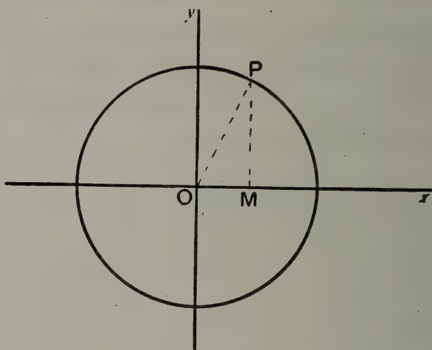

FIG. 32.

Let P (x, y) be any point on the circle of which the centre is O. Let the radius $= a$.

Draw PM $\perp$ Ox. Join PO.

$\because$ OPM is a rt.-$\angle$d $\triangle$,

$\therefore$ $OM^2 + PM^2 = OP^2$.

$\therefore$ $x^2 + y^2 = a^2$.

This being the relation which holds between the coordinates of any point on the circle and the given radius is the required equation.

69. To find the equation of a circle, the centre being at any fixed point (h, k) and the radius equal to a.

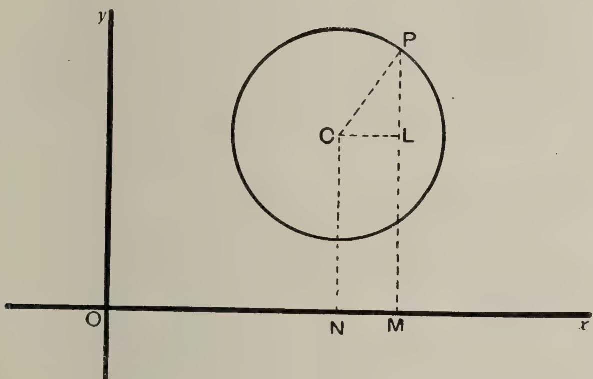

FIG. 33.

C (h, k) is the centre; and P (x, y) is any point on the circle.

Draw PM, CN $\perp$ Ox, CL $\perp$ PM. Join CP.

$$CL = NM = OM - ON = x - h;$$

$$PL = PM - LM = PM - CN = y - k.$$

$\because$ CPL is a rt.-$\angle$d $\triangle$,

$$CL^2 + PL^2 = CP^2.$$

$$\therefore (x - h)^2 + (y - k)^2 = a^2.$$

This is the required equation.

70. If we expand the equation found in § 69, we obtain:—

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0.$$

Comparing this result with the general equation of the second degree:—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we see that the conditions that the latter should represent a circle are that the coefficients of x^2 and y^2 should be equal and that the coefficient of xy should be zero.

Thus the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0$$

may be changed to

$$\left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2 + f^2 - ac}{a^2},$$

from which, by comparison with the formula of § 69, we see that it represents a circle having its centre at the point $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and its radius $= \frac{\sqrt{g^2 + f^2 - ac}}{a}$.

71. The general equation of the circle to rectangular axes is commonly written:—

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0.}$$

When the circle passes through the origin and its centre is on the axis of x, the equation of § 69 becomes

$$(x - a)^2 + y^2 = a^2.$$

or, $$\mathbf{x^2 + y^2 = 2ax.}$$

72.—Exercises

1. Write the equation of the circle with centre (0, 0) and radius $= \sqrt{5}$.

2. Write the equation of the circle with centre (6, 2) and radius $= 3$.

3. Write the equation of the circle with centre $(-5, -1)$ and radius $= \sqrt{26}$. Show that this circle passes through the origin.

4. Write the equation of the circle with centre $(-a, -b)$ and radius $= c$. Find the condition that this circle passes through the origin.

5. Find the coordinates of the centre and the radii of the following circles:—

(*a*), $x^2 + y^2 - 6x - 2y = 15$; (*b*), $4x^2 + 4y^2 + 7x + 5y = 16$; (*c*), $x^2 + y^2 = 14x$; (*d*), $x^2 + y^2 + 2\,by = c^2$.

6. Draw, on squared paper, the circles of which the equations are:—

(*a*), $x^2 + y^2 = 9$; (*b*), $x^2 + y^2 = 8x$; (*c*), $x^2 + y^2 + 6y = 7$.

7. Find the centre and radius of the circle which passes through the origin and cuts off intercepts $= a$ and b from Ox and Oy respectively.

Solution.—Since the circle passes through the origin its equation must be satisfied by $x = 0$, $y = 0$, and $\therefore$ the absolute term must be zero. Thus the equation may be written

$$x^2 + y^2 + 2\,gx + 2fy = 0.$$

Substituting in this equation the coordinates of the points $(a, 0)$, $(0, b)$ the two equations

$$a^2 + 2\,ga = 0$$
$$b^2 + 2\,fb = 0$$

are obtained from which $g = -\dfrac{a}{2}$, $f = -\dfrac{b}{2}$.

$\therefore$ the equation of the circle is

$$x^2 + y^2 - ax - by = 0,$$

$$\text{or, } \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}.$$

$\therefore$ the centre is $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ and the radius $= \dfrac{\sqrt{a^2 + b^2}}{2}$.

8. Find the equation of the circle which passes through the origin and also through (4, 3) and (– 2, 6).

9. Find the equation of the circle which has its centre on the axis of x and which passes through the points (5, 3) and (– 3, 1).

10. Show from the general equation of § 71 that three conditions are necessary and sufficient to determine a circle.

11. Find the condition that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ may have its centre (a) on the axis of x; (b) on the axis of y.

12. Find the equation of the circle which passes through (3, 1) and (5, – 3) and has its centre on the line $x - y = 4$.

13. Find the equation of the circle having the st. line joining (7, – 5) and (– 3, – 1) as a diameter.

14. A (a, 0) is a fixed point and P (x, y) is a variable point such that PO : PA $= p : q$. Show that the locus of P is a circle having its centre on Ox, and dividing OA internally and externally in the ratio $p : q$.

15. Find the equation of the circumcircle of the △ whose vertices are (3, 4), (– 2, 3), (– 5, – 7).

16. Find the length of the chord of the circle $x^2 + y^2 = 25$ cut off by the line $3x + y = 15$.

17. Find the length of the chord of the circle $x^2 + y^2 - 6x + 14y = 42$ cut off by the line $x - y = 8$.

18. Show that the locus of the centres of all circles which pass through two given points (p, q), (r, s) is the right bisector of the st. line joining the given points.

19. Through the given point P (h, k) a st. line is drawn cutting the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at A and B. Prove that PA . PB is constant for all directions of the st. line.

Solution :—Take $\dfrac{x-h}{\cos\theta}=\dfrac{y-k}{\sin\theta}=r$ as the equation of the st. line. Then

$$x = h + r\cos\theta,\ y = k + r\sin\theta.$$

Substituting these values in the equation of the circle, and simplifying

$$r^2 + 2\left\{(h+g)\cos\theta + (k+f)\sin\theta\right\} r$$
$$+ h^2 + k^2 + 2gh + 2fk + c = 0.$$

The value of **PA . PB** = the product of the two values of r in this equation

$$= h^2 + k^2 + 2gh + 2fk + c,$$

an expression which does not contain θ and which is $\therefore$ independent of the direction of the line.

20. Find the equation of that chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is bisected at the point (h, k).

Solution :—(First Method). As in Ex. 19, if we take for the equation of the chord $\dfrac{x-h}{\cos\theta}=\dfrac{y-k}{\sin\theta}=r$, we get

$$r^2 + 2\left\{(h+g)\cos\theta + (k+f)\sin\theta\right\} r$$
$$+ h^2 + k^2 + 2gh + 2fk + c = 0.$$

If the chord is bisected at (h, k) the two values of r got from this equation are equal in value but opposite in sign, and

$$\therefore (h+g)\cos\theta + (k+f)\sin\theta = 0.$$

Multiplying the terms of this equation by the equal fractions $\dfrac{x-h}{\cos\theta}$, $\dfrac{y-k}{\sin\theta}$, the required equation is found to be

$$(h+g)(x-h) + (k+f)(y-k) = 0.$$

(Second Method). Let the equation of the chord be

$$y - k = m(x - h).$$

The centre of the circle is $(-g, -f)$, and the equation of the $\perp$ from the centre to the chord is

$$m(y+f) + x + g = 0.$$

The $\perp$ from the centre bisects the chord, and, $\therefore$ passes through (h, k).

$$\therefore m(k+f) + h + g = 0.$$

$$\therefore m = -\frac{h+g}{k+f}.$$

$\therefore$ the required equation is

$$(h+g)(x-h) + (k+f)(y-k) = 0.$$

21. Find the equation of the chord of the circle $x^2 + y^2 - 6x - 8y = 24$ which passes through $(5, -1)$ and is bisected at that point.

22. From the point P $(-3, -7)$ a st. line is drawn to cut the circle $x^2 + y^2 - 4x - 10y = 17$ at A and B. Find the area of the rectangle PA . PB.

23. Find the equation of the common chord of the circles

$$x^2 + y^2 - 8x - 6y = 39,$$

$$x^2 + y^2 + 6x + 8y = 56.$$

24. Find the condition that the common chord of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

passes through the origin.

25. Find the equation of the circle which passes through the origin and also through (h, k) and (k, h).

Tangents

73. Let APQ be a secant cutting a curve at P and Q.

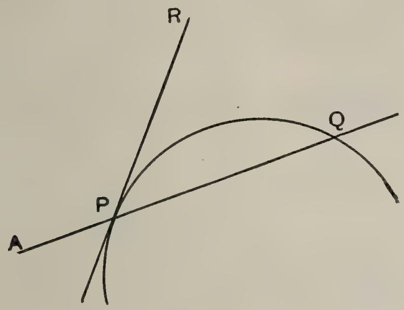

Fig. 34.

If the secant rotate about the point P until the second point Q approaches indefinitely near to P, the limiting position PR of the chord is called a **tangent** to the curve at the point P.

The point P is called the **point of contact** of the tangent PR.

74. To find the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point P (x_1, y_1) on the circle.

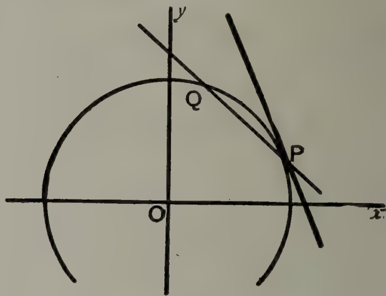

FIG. 35.

Let Q (x_2, y_2) be another point on the circle.

Then the equation of PQ is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \quad (1)$$

$\because$ P and Q are both on the circle,

$$x_1^2 + y_1^2 = a^2,$$

$$\text{and, } x_2^2 + y_2^2 = a^2.$$

$\therefore$, subtracting, $x_1^2 - x_2^2 + y_1^2 - y_2^2 = 0.$

$\therefore$, $(x_1 - x_2)(x_1 + x_2) + (y_1 - y_2)(y_1 + y_2) = 0 \quad (2)$

Multiplying the terms of (2) by the equal fractions in (1)

$$(x - x_1)(x_1 + x_2) + (y - y_1)(y_1 + y_2) = 0. \quad (3)$$

If, now, PQ rotates about P until Q coincides with P, $x_2 = x_1$ and $y_2 = y_1$.

Thus equation (3) becomes

$$2(x - x_1)x_1 + 2(y - y_1)y_1 = 0,$$

$$\text{or,} \quad xx_1 + yy_1 = x_1^2 + y_1^2.$$

But $$x_1^2 + y_1^2 = a^2.$$

$$\therefore\ \mathbf{xx_1 + yy_1 = a^2}.$$

This is the required equation.

75. **Alternative Method** of finding the equation of the tangent at the point P (x_1, y_1) on the circle $x^2 + y^2 = a^2$.

Using the figure of § 74 let the equation of PQ be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r, \qquad (1)$$

and $\therefore\ x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta.$

Substituting these values of x, y in the equation of the circle, and expanding

$$r^2 + 2(x_1\cos\theta + y_1\sin\theta)\,r + x_1^2 + y_1^2 = a^2 \qquad (2)$$

Since P is on the circle, $x_1^2 + y_1^2 = a^2$.

$\therefore$ one value of r is zero, and equation (2) becomes

$$r + 2(x_1\cos\theta + y_1\sin\theta) = 0. \qquad (3)$$

If, now, PQ rotates about P until Q coincides with P, the other value of r also becomes zero, and,

$$\therefore\ x_1\cos\theta + y_1\sin\theta = 0. \qquad (4)$$

Multiplying the terms in (4) by the equal quantities in (1),

$$x_1(x - x_1) + y_1(y - y_1) = 0.$$

$$\therefore\ xx_1 + yy_1 = x_1^2 + y_1^2 = a^2.$$

$\therefore$ the required equation is

$$xx_1 + yy_1 = a^2.$$

76. The equation of OP (Fig. 35) is $\frac{x}{x_1} = \frac{y}{y_1}$, and by the condition of perpendicularity, the line represented by this equation is $\perp$ to the line represented by $xx_1 + yy_1 = a^2$.

$\therefore$ the radius of a circle drawn to the point of contact of a tangent is $\perp$ to the tangent.

77. In any curve, the st. line drawn through the point of contact of a tangent and $\perp$ to the tangent is called a **normal** to the curve at that point.

78. **To find the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a point P (x_1, y_1) on the circle.**

Let the equation of a chord PQ be

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r, \qquad (1)$$

and $\therefore$ $x = x_1 + r \cos \theta$, $y = y_1 + r \sin \theta$.

Substituting these values of x, y in the equation of the circle and simplifying,

$$r^2 + 2 \{ (x_1 + g) \cos \theta + (y_1 + f) \sin \theta \} r + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Since P (x_1, y_1) is a point on the circle, this equation reduces to

$$r + 2 \{ (x_1 + g) \cos \theta + (y_1 + f) \sin \theta \} = 0.$$

If now the secant PQ rotates about P until Q coincides with P, the second value of r becomes zero, and $\therefore$

$$(x_1 + g) \cos \theta + (y_1 + f) \sin \theta = 0. \qquad (2)$$

Multiplying the terms in (2) by the equal quantities in (1),

$$(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0.$$

$$\therefore \quad x(x_1 + g) + y(y_1 + f) - x_1^2 - y_1^2 - gx_1 - fy_1 = 0.$$

But, $\quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$

$\therefore$, adding,

$$x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0.$$

$$\therefore \quad \mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$$

79. By comparing the equation of the tangent

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

with that of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

the following rule is obtained for writing the equation of the tangent at a point (x_1, y_1) on the circle:—

In the equation of the circle change

x^2 into xx_1, $\quad y^2$ into yy_1,

$2x$ " $\quad x + x_1$, $2y$ " $\quad y + y_1$.

80. To find the equation of the tangent to the circle $x^2 + y^2 = a^2$ in terms of m, the slope of the tangent.

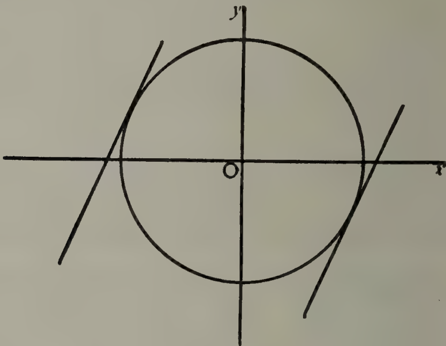

FIG. 36.

To find the abscissae of the points where the line $y = mx + k$ cuts the circle, eliminate y by substitution, and

$$x^2 + (mx + k)^2 = a^2,$$

$$i.e., \ (1 + m^2)\, x^2 + 2\, mkx + k^2 - a^2 = 0.$$

If the line is a tangent, the values of x from this equation are equal to each other, and

$$\therefore \ m^2k^2 = (1 + m^2)\,(k^2 - a^2),$$

$$\therefore \quad k^2 = a^2\,(1 + m^2),$$

$$\text{and } k = \pm\, a\sqrt{1 + m^2}.$$

Thus the equation of the tangent is

$$\mathbf{y = mx \pm a\sqrt{1 + m^2}}.$$

The double sign corresponds to the two tangents that have the same slope, as indicated in the diagram.

81.—Exercises

1. Find the equation of the tangent to the circle

(*a*) $x^2 + y^2 = 34$, at the point $(3, 5)$;

(*b*) $x^2 + y^2 - 10x + 12y = 39$ at $(-1, 2)$;

(*c*) $x^2 + y^2 + 18x - 14y = 39$ at $(3, 12)$;

(*d*) $x^2 + y^2 + 2gx + 2fy = 0$ at the origin.

2. Find the equations of the st. lines touching the circle $x^2 + y^2 = 35$ and making an $\angle$ of $45°$ with the axis of x.

3. Find the equations of the st. lines which touch $x^2 + y^2 = r^2$ and are (*a*) $\parallel$ to, (*b*) $\perp$ to the line $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$

4. Prove that $x + 2y = 10$ is a tangent to the circle $x^2 + y^2 = 20$; and find the point of contact

5. Prove that $x - 2y = 4$ is a tangent to the circle $x^2 + y^2 - 8x - 10y + 21 = 0$; and find the point of contact

6. Find the condition that $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ may touch (*a*) $x^2 + y^2 = r^2$; (*b*) $x^2 + y^2 + 2gx + 2fy + c = 0$.

7. Find the condition that $\frac{x}{a} + \frac{y}{b} = 1$ may touch $x^2 + y^2 = r^2$.

8. Find the condition that the axis of x may touch $x^2 + y^2 + 2gx + 2fy + c = 0$.

9. Find the equations of the circles passing through $(5, 2)$ and touching the axes of x and y.

10. Find the equations of the tangents to the circle $x^2 + y^2 - 2x + 2y = 10$ which make an $\angle = 30°$ with the axis of x.

11. Find the length of the part of the line $2x - 5y + 10 = 0$ intercepted by the circle $x^2 + y^2 + 8x - 6y = 24$.

12. Show that the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point (x_1, y_1) on this circle is $(x - a)(x_1 - a) + (y - b)(y_1 - b) = r^2$.

Solution.—Transform the origin to the point (a, b) without changing the direction of the axes. The transforming relations are $x = X + a$, $y = Y + b$, $x_1 = X_1 + a$, $y_1 = Y_1 + b$.

The equation of the circle becomes

$$X^2 + Y^2 = r^2,$$

and, by § 74, the tangent at (X_1, Y_1) is

$$XX_1 + YY_1 = r^2.$$

Transforming back to the original origin, the equation of the tangent becomes

$$(x - a)(x_1 - a) + (y - b)(y_1 - b) = r^2.$$

13 Show that the point $(g + r \cos a, f + r \sin a)$ is on the circle $(x - g)^2 + (y - f)^2 = r^2$; and find the equation of the tangent at that point.

14. Show that the circles

$$x^2 + y^2 + 6x + 16y + 24 = 0,$$

$$x^2 + y^2 - 10x + 4y + 20 = 0$$

touch each other externally. Find the coordinates of the point of contact; and the equation of the common tangent at that point.

15. Show that, if the circles

$$(x - h)^2 + (y - k)^2 = r^2,$$

$$(x - m)^2 + (y - n)^2 = s^2$$

touch each other,

$$(h - m)^2 + (k - n)^2 = (r \pm s)^2.$$

16. Find the equation of the common chord; and the coordinates of the points of intersection of the circles

$$x^2 + y^2 - 2x - 6y = 14,$$
$$x^2 + y^2 - 5x + 3y = 5.$$

17. Find the equation of the circle whose centre is at the origin and which touches the line $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$.

18. Find the equation of the circle whose centre is at $(7, 2)$ and which touches $3x - 5y = 4$.

19. Find the centres of similitude and the equations of the transverse and direct common tangents of the circles

$$x^2 + y^2 - 6x + 2y + 6 = 0,$$
$$x^2 + y^2 + 8x - 10y + 32 = 0.$$

20. Find the equations of the common tangents of the circles

$$x^2 + y^2 - 4x - 8y - 5 = 0,$$
$$x^2 + y^2 - 10x - 6y - 2 = 0.$$

Find also the coordinates of the points of contact of the tangents.

21. Find the equations of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which are $\parallel$ to $x + 3y = 9$.

22. Find the equations of the two tangents to the circle $x^2 + y^2 = 25$ which make an $\angle$ of $30°$ with the axis of x.

Poles and Polars

82. To find the equation of the chord of contact of tangents drawn from an outside point to the circle $x^2 + y^2 = a^2$.

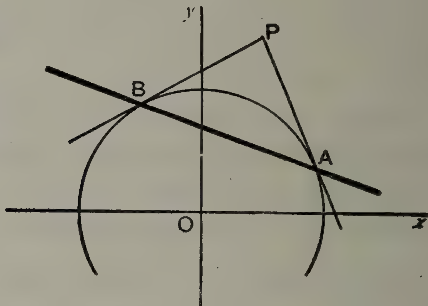

Fig. 37.

Let P (x_1, y_1) be the given point; PA and PB the tangents.

It is required to find the equation of AB.

Take (x', y'), (x'', y'') to represent the coordinates of A, B respectively.

The equation of AP is, by § 74,

$$xx' + yy' = a^2;$$

and since the coordinates of P must satisfy this equation,

$$x_1x' + y_1y = a^2. \tag{1}$$

Similarly, $$x_1x'' + y_1y'' = a^2. \tag{2}$$

From these results it is seen that

$$xx_1 + yy_1 = a^2$$

is the equation of **AB**; for:—

since it is of the first degree it represents a st. line;

by (1), **A** is a point on the line;

by (2), **B** is a point on the line;

∴ the required equation is

$$\mathbf{xx_1 + yy_1 = a^2}.$$

83. The equation of the chord of contact of tangents drawn from an outside point to a circle is of the same form as the equation of the tangent at a point on the circle.

This is in agreement with the fact that, if the point **P** approach the circle and ultimately fall on it, the chord of contact becomes the tangent at **P**, or the tangent at **P** is the final position of the chord of contact when **P** approaches the circle.

84. To find the equation of the polar of P (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$.

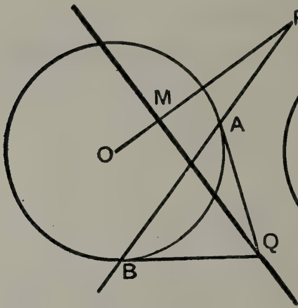

FIG. 38.

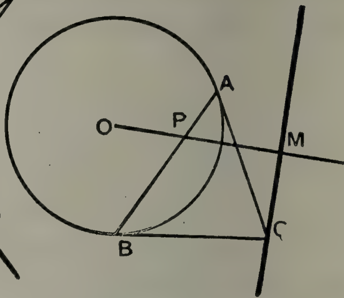

FIG. 39.

Through P draw any st. line cutting the circle at A, B. Draw tangents AQ, BQ intersecting at Q (X, Y).

It is required to find the locus of Q.

By § 82, the equation of AB is

$$x\mathrm{X} + y\mathrm{Y} = a^2;$$

and as the coordinates of P must satisfy this equation

$$\mathrm{X}x_1 + \mathrm{Y}y_1 = a^2.$$

$\therefore$, as X, Y are the coordinates of any point on the polar of P, the required equation is

$$\mathbf{xx_1 + yy_1 = a^2}.$$

85. The equation of OP is $xy_1 - yx_1 = 0$, and, by the condition for perpendicularity, this line is $\perp$ to that represented by $xx_1 + yy_1 = a^2$.

$\therefore$ the polar of P is a st. line which cuts OP at rt. $\angle$s

86. If the polar $xx_1 + yy_1 = a^2$ cuts OP at M, the length of OM $= \dfrac{a^2}{\sqrt{x_1^2 + y_1^2}} = \dfrac{a^2}{\text{OP}}$.

$$\therefore \text{ OM . OP} = a^2.$$

87. The equation of the polar of the point P (x_1, y_1) without the circle $x^2 + y^2 = a^2$ is the same as that of the chord of contact of tangents drawn from P to the circle. This shows that, when the point is without the circle, its polar is the chord of contact produced, or, that **tangents drawn from P touch the circle at the points where it is cut by the polar of P.**

88. The equation of the polar of any point P (x_1, y_1) is of the same form as the equation of the tangent at a point on the circle.

This is in agreement with the fact that, if the point P approaches and ultimately coincides with the circle, OP becomes equal to a, and $\therefore$, by § 86, OM becomes equal to a, and the polar becomes the tangent at the point P

89. **If the polar of P passes through Q, the polar of Q passes through P.**

Let (x_1, y_1), (x_2, y_2) be the coordinates of P, Q respectively.

The polar of P with respect to $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2.$$

Since this line passes through Q,

$$x_2x_1 + y_2y_1 = a^2.$$

This proves that (x_1, y_1) is on the line

$$xx_2 + yy_2 = a^2;$$

and $\therefore$ P is on the polar of Q.

Cor. If the point Q moves along the polar of P, the polar of Q changes its position, but always passes through P.

$\therefore$, if the pole moves along a st. line, its polar turns about the pole of that line.

90. **To find the pole of the st. line $Ax + By + C = 0$ with respect to the circle $x^2 + y^2 = a^2$.**

Let (x_1, y_1) be the coordinates of the pole.

The equation of the polar of (x_1, y_1) is

$$xx_1 + yy_1 - a^2 = 0.$$

This equation must be the same as

$$Ax + By + C = 0.$$

$$\therefore \qquad \frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C}$$

$$\therefore \quad x_1 = -\frac{a^2A}{C}, \; y_1 = -\frac{a^2B}{C}.$$

91. To find the polar of P (x_1, y_1) with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

The equation of the circle may be written

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

Transforming the origin to the point $(-g, -f)$, the transforming relations are $x = X - g$, $y = Y - f$, $x_1 = X_1 - g$, $y_1 = Y_1 - f$, and the equation of the circle becomes

$$X^2 + Y^2 = g^2 + f^2 - c.$$

The equation of the polar of P (X_1, Y_1) with respect to this circle is, by § 84,

$$XX_1 + YY_1 = g^2 + f^2 - c.$$

Transforming back to the original origin, the equation becomes

$$(x + g)(x_1 + g) + (y + f)(y_1 + f) = g^2 + f^2 - c.$$

$$\therefore \mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$$

NOTE.—*As an exercise, the student should obtain the above result directly from the definition of poles and polars, by the method used in § 84.*

92.—Exercises

1. Find the polar of the point

(*a*) (3, 5) with respect to $x^2 + y^2 = 30$;

(*b*) $(a, 0)$ " " " $x^2 + y^2 = r^2$;

(*c*) $(-2, 4)$ " " " $x^2 + y^2 - 4x - 8y = 5$;

(*d*) $(-5, -1)$ " " " $x^2 + y^2 - 10x + 6y = 15$;

(*e*) (0, 0) " " " $(x - h)^2 + (y - k)^2 = r^2$.

2. Find the pole of the st. line

(*a*) $2x - 7y = 17$ with respect to $x^2 + y^2 = 17$;

(*b*) $x - 2y + 12 = 0$ " " " $x^2 + y^2 = 23$;

(*c*) $4x - y = 1$ with respect to $x^2 + y^2 - 2x - 4y = 4$;

(*d*) $4x + 5y = 5$ " " " $x^2 + y^2 - 8x - 10y = -5$.

3. (*a*) Show that $x^2 + y^2 = 25$ is the equation of a circle.

(*b*) Show that $(-3, 4)$ is on the circle.

(*c*) Write the equation of the tangent to the circle at this point.

(*d*) Show that the point (9, 13) is on this tangent.

(*e*) Write the equation of the polar of (9, 13).

(*f*) Find the equation of the st. line through (9, 13) $\perp$ to the polar, commenting on the form of the result.

(*g*) Find the equation of the other tangent from (9, 13).

Draw the diagram on squared paper.

4. Find the pole of $\dfrac{x}{a} + \dfrac{y}{b} = 1$ with respect to $x^2 + y^2 = c^2$.

5. Find the pole of $lx + my = 1$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

6. Prove that the polar of $(-2, 5)$ with respect to $x^2 + y^2 = 18$ touches $x^2 + y^2 - 6x + 2y = 19$; and find the coordinates of the point of contact.

TANGENTS FROM AN OUTSIDE POINT

93. To find the length of the tangent PA from the point P (x_1, y_1) to a given circle.

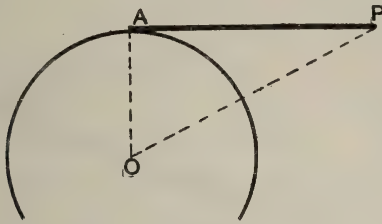

FIG. 40.

(1) Let the equation of the circle be

$$x^2 + y^2 = a^2.$$

Join OA, OP.

$$\because \quad AOP \text{ is a rt.-}\angle\text{d } \triangle,$$

$$AP^2 = OP^2 - AO^2$$

$$= x_1^2 + y_1^2 - a^2.$$

$$\therefore \quad \mathbf{AP = \sqrt{x_1^2 + y_1^2 - a^2}.}$$

(2) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This equation may be written

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c,$$

from which it is seen that the centre is $(-g, -f)$ and the radius $= \sqrt{g^2 + f^2 - c}$.

With the diagram and construction of Fig. 40,

$$AP^2 = OP^2 - AO^2$$

$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$$= x_1^2 + y_1^2 + 2\,gx_1 + 2fy_1 + c.$$

$$\therefore \quad \mathbf{AP = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.}$$

94. To find the equation of the tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$.

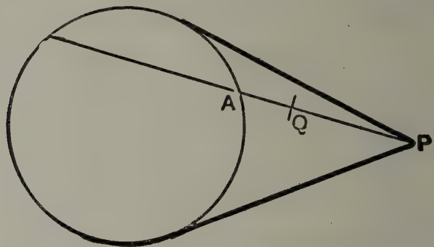

FIG. 41.

Let a secant drawn from P (x_1, y_1) cut the circle at A; and let Q (x, y) be any point on the secant.

If PA : QA $= k : 1$, the coordinates of A are $\left(\frac{kx - x_1}{k - 1}, \frac{ky - y_1}{k - 1}\right)$, and $\therefore$, since A is on the circle

$$(kx - x_1)^2 + (ky - y_1)^2 = (k - 1)^2 a^2,$$

or, $$(x^2 + y^2 - a^2) k^2 - 2 (xx_1 + yy_1 - a^2)k + x_1^2 + y_1^2 - a^2 = 0.$$

If, now, the secant turn about P until it coincides with either of the tangents from P, the two values of k found from this equation, and which correspond to the two points where the secant cuts the circle, are equal to each other.

$$\therefore\ (\mathbf{xx_1} + \mathbf{yy_1} - \mathbf{a}^2)^2 = (\mathbf{x}^2 + \mathbf{y}^2 - \mathbf{a}^2)(\mathbf{x_1}^2 + \mathbf{y_1}^2 - \mathbf{a}^2).$$

This is the required equation.

95.—Exercises

1. Find the length of the tangent from
   (*a*), (7, 3) to $x^2 + y^2 = 22$;
   (*b*), (3, − 5) to $x^2 + y^2 - 3x + 7y + 35 = 0$;
   (*c*), (− 2, − 6) to $x^2 + y^2 = 12x$;
   (*d*), (0, 0) to $x^2 + y^2 + 2gx + 2fy + c = 0$;
   (*e*), (4, 2) to $x^2 + y^2 - 6x + 2y - 39 = 0$.

   Explain the imaginary result in (*e*).

2. The length of the tangent drawn from a point to $x^2 + y^2 - 10x - 4y + 9 = 0$ is always 4. Find the locus of the point. Plot the diagram on squared paper.

3. The length of the tangent from P to $x^2 + y^2 = 9$ is twice the distance from P to (6, 0). Find the locus of P.

4. Find the equations of the tangents from (7, − 1) to $x^2 + y^2 = 25$.

5. Show, by the method of § 94, that the equation of the tangents from (x_1, y_1) to $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\}^2$$
$$= (x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c).$$

Radical Axis

96. To find the radical axis of the circles

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0,}$$
$$\mathbf{x^2 + y^2 + 2g'x + 2f'y + c' = 0.}$$

Since the tangents to the circles from any point on their radical axis are equal to each other, if (x, y) is any point on the locus, by § 93,

$$x^2 + y^2 + 2gx + 2fy + c = x^2 + y^2 + 2g'x + 2f'y + c'$$

∴ the required equation is

$$\mathbf{2(g - g')x + 2(f - f')y + c - c' = 0.}$$

97. The centres of the circles in the last article are $(-g, -f)$ and $(-g', -f')$.

$\therefore$ the st. line joining the centres is

$$\frac{x+g}{g-g'} = \frac{y+f}{f-f'},$$

or, $(f - f')(x + g) - (g - g')(y + f) = 0$.

By the condition of perpendicularity this line is $\perp$ to

$2(g - g')x + 2(f - f')y + c - c' = 0$.

$\therefore$ the radical axis is $\perp$ to the line of centres.

98.—Exercises

1. Find the radical axis of the circles

$$x^2 + y^2 - 4x - 6y + 9 = 0,$$
$$x^2 + y^2 - 16x - 14y + 104 = 0.$$

Draw the diagram on squared paper.

2. Find the radical axis of the circles

$$2x^2 + 2^2y + 9x - 8y - 3 = 0,$$
$$x^2 + y^2 = 9.$$

3. Show that the radical axes of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

taken two and two are concurrent. (The point of concurrence is the radical centre.)

4. Find the radical centre of the circles

$$x^2 + y^2 - 3x + 7y + 35 = 0,$$
$$x^2 + y^2 - 7x + 5y - 31 = 0,$$
$$x^2 + y^2 - 6x + 2y - 39 = 0.$$

5. Show that the circles

$$x^2 + y^2 - 3x + 5y - 9 = 0,$$
$$x^2 + y^2 + 7x + y - 11 = 0,$$
$$x^2 + y^2 + 2x + 3y - 10 = 0,$$

have a common radical axis. Show also that their centres are in a st. line which is ⊥ to the common radical axis.

Miscellaneous Exercises

(a)

1. Find the equation of the st. line passing through the intersection of $x - 2y = 5$, $x + 3y = 10$ and $\parallel$ to $3x + 4y = 11$.

2. Find the equation of the st. line passing through the intersection of $8x + y = 7$, $11x + 2y = 28$ and ⊥ to the latter line.

3. Plot the quadrilateral $(4, 2)$, $(-5, 6)$, $(-9, -6)$, $(7, -4)$; and find its area.

4. Plot the lines $2x + 5y = 29$, $12x + y = 29$, $5x - 2y = 29$; and find the area contained by them.

5. Find the equation of the st. line passing through (h, k) and such that the portion of it between the axes is bisected at the given point.

6. Find the equation of the st. line passing through (h, k) and (a) $\parallel$ to, (b) ⊥ to the st. line joining (x_1, y_1), (x_2, y_2).

7. Show that, if the lines $ax + by + c = 0$, $bx + cy + a = 0$, $cx + ay + b = 0$ are concurrent but not coincident, then $a + b + c = 0$.

8. Find the ratio in which the st. line joining $(-5, 3)$, $(6, -1)$ is divided by $x - 11y + 3 = 0$.

9. Find the centre of the inscribed circle of the $\triangle$ formed by the lines $4x - 3y = 18$, $5x + 12y = 9$, $24x + 7y = 30$.

10. Find the area of the $\triangle$ contained by $y = 3x$, $y = 5x$ and $x + 2y = 77$.

11. Show that the area of the $\triangle$ contained by $y = m_1x$, $y = m_2x$ and $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$

$$= \frac{(m_1 - m_2)\,\mathsf{C}^2}{2\,(\mathsf{A} + m_1\mathsf{B})\,(\mathsf{A} + m_2\mathsf{B})}.$$

12. Find the locus of a point such that the square of its distance from $(6, 0)$ is three times the square of its distance from $(2, 0)$.

13. One vertex of a ||gm is at the origin and the two adjacent vertices are at (a, b), (c, d). Find the fourth vertex.

14. Show that, if the two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 - 2fx - 2gy + c = 0$$

touch each other, then $(g - f)^2 = 2c$.

15. Give the geometrical interpretation of the equation $x^2 + y^2 + 2ax \cos \alpha + 2ay \sin \alpha + a^2 = 0$.

16. Find the locus of the intersection of the st. lines which pass through $(6, 0)$ and $(0, 3)$ respectively and cut each other at rt. $\angle$s.

17. Find the equations of the tangents to the circle $x^2 + y^2 = 2kx$ which are || to $3x - y = 0$.

18. Find the orthocentre of the $\triangle$ whose sides are $8x - 5y = 16$, $2x - 3y = 10$ and $x + 2y = 6$.

19. Prove that the radical axis of the circles

$$x^2 + y^2 = a^2,$$
$$x^2 + y^2 - 2a\,(x \cos \alpha + y \sin \alpha) = 0$$

bisects the st. line joining their centres.

20. Chords of the circle $x^2 + y^2 = a^2$ pass through the fixed point (h, o). Find the locus of their middle points.

21. Find the equation of the circle which passes through the origin and makes intercepts a and b on Ox, Oy respectively.

22. Find the equation of the circle described on the st. line joining the origin to (g, f) as diameter.

23. Find the coordinates of a point such that the st. line joining it to $(4, -3)$ is bisected at rt. $\angle$s by $2x - 3y = 7$.

24. Find the locus of the points from which tangents drawn to $x^2 + y^2 = 13$ and $x^2 + y^2 - 2x + 6y + 1 = 0$ are as 5 is to 3.

25. Find the distances from the point $(2, 4)$ in the direction having the direction cosines $-\frac{3}{5}$, $-\frac{4}{5}$ to the curve whose equation is

$$3x^2 - 6xy + 5y^2 - 32 = 0.$$

26. Find the equation of the locus of a point P such that PA : PB $= k : 1$ where A (x_1, y_1), B (x_2, y_2) are fixed points.

Show that the locus is a circle and find the relation of its centre to A and B.

27. (*a*) Find the coordinates of the point C which divides the st. line joining A $(3, -2)$, B $(19, 10)$ in the ratio AC : CB $= 1 : 3$.

(*b*) Prove that D $(11, 4)$ lies on the st. line AB given above; and by computing the lengths of AD and BD, find the ratio in which D divides AB.

28. (*a*) Find the area of a $\triangle$ the coordinates of whose angular points are (x_1, y_1), (x_2, y_2), (x_3, y_3).

(*b*) From the result of (*a*) deduce the equation of a st. line in terms of the coordinates of two given points through which it passes.

29. Show that $y = mx + a\sqrt{1 + m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$.

30. The equation $3x^2 + 3y^2 - 12x - 6y + 4 = 0$ can be reduced to one containing terms in x and y of the second degree only, by transforming to $\parallel$ axes through a properly chosen point. What are the coordinates of the point?

31. Find the distance of the point of intersection of the lines $3x + 2y + 4 = 0$ and $2x + 5y + 8 = 0$ from the line $5x - 12y + 6 = 0$.

32. Find the equation of the circle whose centre is (h, k) and which passes through (a, b).

33. Find the locus of the points from which tangents drawn to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

are at rt. $\angle$s to each other.

34. If the tangents at the points (x_1, y_1), (x_2, y_2) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at rt. $\angle$s to each other, show that

$$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + g^2 + f^2 = 0.$$

35. Find the equation of the st. line joining $(ab^2, 2ab)$ and $(ac^2, 2ac)$.

36. Show that the equation of the $\perp$ to $\dfrac{\cos a}{a}x + \dfrac{\sin a}{b}y = 1$ at the point $(a \cos a, b \sin a)$ is $\dfrac{a}{\cos a}x - \dfrac{b}{\sin a}y = a^2 - b^2$.

37. Find the product of the $\perp$s from $(-7, -4)$ to the lines $3x^2 - 12xy + 11y^2 = 0$.

38. Show that the product of the $\perp$s from (c, d) to the lines $ax^2 + 2\ hxy + by^2 = 0$ is

$$\frac{ac^2 + 2hcd + bd^2}{\sqrt{(a - b)^2 + 4h^2}}.$$

(b)

39. Find the equation of the st. lines which join the origin to the points of intersection of

$$ax + by = k \qquad (1)$$

$$\text{and} \quad x^2 + y^2 + 2gx + 2fy + c = 0. \qquad (2)$$

Solution:—

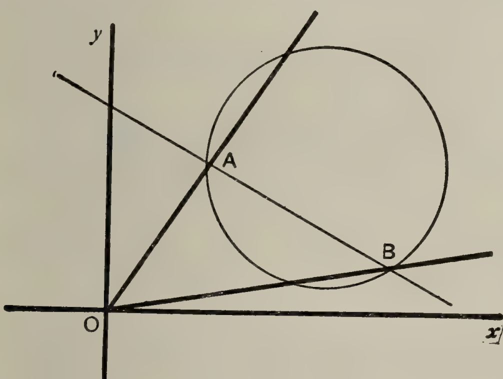

FIG. 42.

Let the line (1) cut the circle (2) at **A, B.**

It is required to find the equation representing **OA** and **OB.**

From (1) $k^2 = k\ (ax + by) = (ax + by)^2$.

$$\therefore \quad k^2(x^2 + y^2) + 2k\ (ax + by)\ (gx + fy) + c\ (ax + by)^2 = 0. \qquad (3)$$

Equation (3) has all its terms of the second degree in x and y, and $\therefore$, by § 56, it represents two st. lines passing through the origin.

Again, equation (3) is satisfied by the values of x and y which satisfy both (1) and (2);

$\therefore$ the lines represented by (3) pass through **A** and **B.**

$\therefore$ equation (3) represents **OA** and **OB.**

40. Find the equation of the st. lines joining the origin to the points of intersection of $2x - 3y = 1$ and $x^2 + y^2 = 5$.

41. Find the equation of the st. lines joining the origin to the points of intersection of $x + 2y = 2a$ and $5(x^2 + y^2) + 5ax + 10ay = 18a^2$, and show that they are $\perp$ to each other.

42. Find the $\angle$ between the st. lines which join the origin to the points of intersection of $\frac{x}{2} - \frac{y}{3} = 1$ and $x^2 + y^2 - 2x + 6y + 1 = 0$.

43. Find the equations of the st. lines passing through the intersection of $3x + 2y = 7$ and $x + 5y = 11$ and such that the $\perp$ on each of them from $(4, 7)$ is equal to 5.

44. A, B are points on Ox, Ox' respectively and on OA, OB squares OACD, OBEF are described. EF produced cuts AC at G. Prove that OG, BC, ED are concurrent.

45. If $\frac{1}{a} + \frac{1}{b}$ is constant, show that the variable line $\frac{x}{a} + \frac{y}{b} = 1$ passes through a fixed point.

46. A st. line moves so that the sum of the $\perp$s to it from (a, b), (c, d) is equal to the $\perp$ to it from (g, h). Show that the st. line passes through a fixed point and find the coordinates of the point.

47. Prove that the difference of the squares of the tangents from (x_1, y_1) to the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is equal to twice the rectangle contained by the distance between the centres of the circles and the length of the $\perp$ from (x_1, y_1) to their radical axis.

48. Three circles touch each other at a common point. Prove that the polars of a fixed point (x_1, y_1) with respect to these circles are concurrent.

49. Find the equations of the st. lines which divide the $\angle$s between the lines $4x - 3y + 7 = 0$, $5x + 12y - 19 = 0$ into parts whose sines are as 5 to 7.

50. Show that the equation of the st. line joining $\{a \cos(\alpha + \beta), b \sin(\alpha + \beta)\}$ and $\{a \cos(\alpha - \beta), b \sin(\alpha - \beta)\}$ is $\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = \cos \beta$.

51. Show that the bisectors of the interior $\angle$s of a $\triangle$ are concurrent.

NOTE.—*Take the origin within the $\triangle$, and let the equations of the sides be*

$$x \cos \alpha_1 + y \sin \alpha_1 = p_1,$$
$$x \cos \alpha_2 + y \sin \alpha_2 = p_2,$$
$$x \cos \alpha_3 + y \sin \alpha_3 = p_3.$$

52. If the chord of the circle $x^2 + y^2 = a^2$ whose equation is $px + qy = 1$ subtends an $\angle$ of $45°$ at the origin, then $a^2(p^2 + q^2) = 4 - 2\sqrt{2}$.

53. A st. line moves so that the sum, or the difference, of the intercepts cut off from the axes varies as the area of the $\triangle$ contained by the st. line and the axes. Prove that the st. line passes, in either case, through a fixed point.

54. Show that the area of the $\triangle$ contained by the lines $ax^2 + 2hxy + by^2 = 0$ and $Ax + By + C = 0$ is

$$\frac{C^2 \sqrt{h^2 - ab}}{A^2 b - 2ABh + B^2 a}.$$

55. OACB is a $\|$gm, P is a point in OA, Q is a point in OB; PS drawn $\|$ OB meets BC at S; QR drawn $\|$ OA meets AC at R. Show that PR, QS, OC are concurrent.

56. P is a point such that the sum of the $\perp$s from P on Ox and on $x - by = 0$ is constant. Prove that the locus of P is the base of an isosceles $\triangle$ of which O is the vertex and $y = 0$, $x - by = 0$ are the sides.

57. Given the base of a $\triangle$ in magnitude and position and the magnitude of its vertical $\angle$; prove that the locus of its vertex is a circle.

58. Prove that, if (x_1, y_1), (x_2, y_2) are the extremities of the diameter of a circle, the equation of the circle may be written

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

59. If (h, k) is a point in the first quadrant, show that the equation of the st. line which passes through (h, k) and makes with the axes in that quadrant the $\triangle$ of minimum area is $\frac{x}{h} + \frac{y}{k} = 2$.

60. Show that, if the chord of contact of tangents drawn from the point (h, k) to the circle $x^2 + y^2 = r^2$ subtends a rt. $\angle$ at the centre, then $h^2 + k^2 = 2r^2$.

61. P, Q are two points and O is the centre of a circle. PM is $\perp$ to the polar of Q with respect to the circle, and QN is $\perp$ to the polar of P. Show that PM : QN = OP : OQ.

62. Tangents PA, PB are drawn from the point P (h, k) to the circle $x^2 + y^2 = r^2$. Prove that

$$\triangle \text{PAB} = \frac{r(h^2 + k^2 - r^2)^{\frac{3}{2}}}{h^2 + k^2}.$$

63. Prove that the polar of (a, b) with respect to $x^2 + y^2 = c^2$ is a tangent to $(x - h)^2 + (y - k)^2 = r^2$, if $(ah + bk - c^2)^2 = (a^2 + b^2)r^2$.

64. **ABC** is a $\triangle$ in which a variable line **DE** drawn $\parallel$ to **BC** cuts **AB** at **D** and **AC** at **E**. Show that the locus of the intersection of **BE**, **CD** is the st. line joining **A** to the middle point of **BC**.

65. Show that the equation of the system of circles which pass through (h, k) and touch $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ may be written

$$(\mathrm{A}h + \mathrm{B}k + \mathrm{C})\left\{(\mathrm{A}x + \mathrm{B}y + \mathrm{C})^2 + (\mathrm{B}x - \mathrm{A}y + l)^2\right\}$$
$$= (\mathrm{A}x + \mathrm{B}y + \mathrm{C})\left\{(\mathrm{A}h + \mathrm{B}k + \mathrm{C})^2 + (\mathrm{B}h - \mathrm{A}k + l)^2\right\},$$

where l is an arbitrary constant.

66. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts off from $\mathrm{O}x$, $\mathrm{O}y$ chords of which the lengths are respectively a and b. Show that $4g^2 - a^2 = 4f^2 - b^2 = 4c$.

67. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which pass through the fixed point (h, k).

68. **O** is the centre of a fixed circle, **A** is a fixed point, **Q** is any point on the circle. The bisector of $\angle$ **AOQ** meets **AQ** at **P**. Show that the locus of **P** is a circle having its centre in **AO**.

69. Find the equation of the circle with its centre on $\mathrm{O}x$ and which cuts $x^2 + y^2 = 9$ and $5(x^2 + y^2) = 9x$ orthogonally.

70. Show that the circles $x^2 + y^2 + 2x - 2y = 23$ and $7(x^2 + y^2) - 192x + 144y = 175$ cut orthogonally at the point $(3, 4)$.

71. If the chord of the circle $x^2 + y^2 = r^2$ on the line $px + qy = 1$ subtends an $\angle$ of $45°$ at the origin, then

$$\left\{r^2(p^2 + q^2) - 4\right\}^2 = 8.$$

72. A rt. $\angle$ is subtended at the origin by the chord of the circle $(x - h)^2 + (y - k)^2 = r^2$ on the line $x \cos \alpha + y \sin \alpha = p$. Show that $2p^2 - 2p(h \cos \alpha + k \sin \alpha) + h^2 + k^2 = r^2$.

73. Show that the condition that the circles $x^2 + y^2 = r^2$, $x^2 + y^2 + 2gx + 2fy + c = 0$ touch each other is

$$(r^2 + c)^2 = 4r^2(g^2 + f^2).$$

74. Find the $\angle$ between the tangents at a point of intersection of the circles $x^2 + y^2 - 4x - 8y = 5$ and $x^2 + y^2 - 10x - 6y = 2$.

75. The equal sides **OA**, **OB** of an isosceles rt.-$\angle$d $\triangle$ are produced to **P**, **Q** such that **AP . BQ = OA²**. Show that **PQ** passes through a fixed point.

76. Find the locus of a point such that a tangent drawn from it to the circle $x^2 + y^2 - 8x - 10y = 8$ is twice a tangent drawn from it to $x^2 + y^2 = 25$.

77. Find the equation of the circle which passes through the points of intersection of $x^2 + y^2 = a^2$, $x^2 + y^2 = 2a(x + y)$, and touches the line $x + y = 2a$.

78. Show that, if the line $\dfrac{x - h}{\cos \theta} = \dfrac{y - k}{\sin \theta} = r$ cuts the circle $x^2 + y^2 = a^2$ at **D**, **E** and the polar of the point **P** (h, k) with respect to the circle at **F**, then **P**, **D**, **F**, **E** is a harmonic range.

What is the general statement of this proposition?

79. If $axy + bx + cy + d = 0$ represents two st. lines, show that the lines intersect at the point $\left(-\dfrac{c}{a}, -\dfrac{b}{a}\right)$.

80. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two st. lines, show that the squares of the coordinates of the intersection of the lines are $\dfrac{f^2 - bc}{h^2 - ab}$ and $\dfrac{g^2 - ac}{h^2 - ab}$.

81. The st. lines $y + mx = b$, $y + mx = c$ cut the axes at **A**, **B** and **A′**, **B′** respectively. Find the area of **AA′B′B**.

82. Show that the polar of (k, h) with respect to the circle which has its centre at $(h\ k)$ and touches the line $lx + my + 1 = 0$ is $(l^2 + m^2)(h - k)(y - x + h - k) = (lh + mk + 1)^2$.

83. Prove that the locus of the middle point of the chord of contact of tangents drawn from points on a given st. line to a given circle is a circle passing through the centre of the given circle and having its centre on the $\perp$ from the centre of the given circle to the given st. line.

84. Three concentric circles, **A**, **B**, **C**, have their radii in G. P. Show that, if the pole with respect to **B** of a st. line is on **A**, the polar will touch **C**; and if the pole is on **C**, the polar will touch **A**.

85. Find the equation of the bisectors of the angles contained by the lines $x^2 + y^2 + kxy = 0$.

86. Find the locus of a point such that the $\perp$ from it to the line $x + y = a$ is the geometrical mean between the coordinates of the point.

87. Show that the circles $x^2 + y^2 + 2gx + 2fy = 0$, $x^2 + y^2 + 2fx - 2gy = 0$ cut orthogonally.

88. Find the equation of the circle which cuts orthogonally each of the circles:—

$$x^2 + y^2 + x + 2y = 3, \quad x^2 + y^2 + 3x + 4y = 7, \quad x^2 + y^2 + 4x + 4y = 8.$$

89. Points **A**, **B** are given in **O**x; **C**, **D** in **O**y such that **OA**, **OB**, **OC**, **OD** are in H. P. Show that the locus of the intersection of **AD** and **BC** is $x = y$.

90. Show that the lines $x + 13y = 0$, $3x = 5y$, $5x = 7y$, and $7x = 8y$ form a harmonic pencil.

91. The circle $x^2 + y^2 = r^2$ cuts **O**x', **O**x at **C**, **D** respectively. **EF** is a chord such that $\angle$ **EOF** $= 2a$, and **CE**, **DF** intersect at **P**. Show that the locus of **P** is the circle

$$x^2 + y^2 - 2ry \tan a = r^2.$$

92. Find the equation of a circle passing through the intersections of the circles:—

$$(1)\ x^2 + y^2 - 4x - 8y = 28,\quad (2)\ x^2 + y^2 = 9;$$

and through the centre of (1).

93. Show that the points (1, 7), (− 2, 8), (3, 3) and (2, 6) are concyclic.

94. From any point **A** in the line $x = y$ st. lines are drawn making $\angle$s of 60° and 120° with **O**x and cutting y'**O**y at **B** and **C** respectively. From **OC** a part **OD** is cut off = **OB**. Show that **CD** = the diagonal of a square on **OA**.

95. **D**, **E** are respectively points in two given st. lines **OX**, **OY** such that **OD** + **OE** = c; and **P** is a point in **DE** such that **DP** = m. **EP**. If **OX**, **OY** are taken as axes of coordinates, show that the locus of **P** is $(m + 1)(mx + y) =$ cm.

96. A point **P** moves such that the distance of **P** from a fixed point equals the tangent from **P** to a fixed circle. Show that the locus of **P** is a st. line $\perp$ to the st. line joining the fixed point to the centre of the circle.

97. Show that the st. lines represented by $bx^2 - 2hxy + ay^2 = 0$ are respectively $\perp$ to the st. lines represented by $ax^2 + 2hxy + by^2 = 0$.

98. A series of circles touch the axis of x at the origin. Show that the tangents at the points where the line $y = b$ cuts the circles all touch the fixed circle $x^2 + y^2 = b^2$.

99. A circle of given radius moves so that its radical axis with reference to a fixed circle always passes through a fixed point. Show that the locus of its centre is a circle having its centre at the fixed point.

100. Show that the quadrilateral enclosed by the lines $3x + 2y = 0$, $2x - 3y + 1 = 0$, $2x - 3y = 0$, $3x + 2y = 1$ is a square.

101. Show that the centre of the circle
$x^2 + y^2 - 2gx - 2fy + c + l\,(x^2 + y^2 - 2hx - 2ky + d) = 0$ divides the st. line joining the centres of the circles $x^2 + y^2 - 2gx - 2fy + c = 0$ and $x^2 + y^2 - 2hx - 2ky + d = 0$ in the ratio of $l : 1$.

102. The sum of the $\perp$s from two fixed points (x_1, y_1) and (x_2, y_2) to a variable line $lx + my + n = 0$ is equal to the constant a. Prove that the line is always tangent to a fixed circle, and find the equation of the circle. (Problems—1907.)

103. Show that the circles $x^2 + y^2 + 2x - 8y + 8 = 0$, $x^2 + y^2 + 10x - 2y + 22 = 0$ touch each other. (Problems—1911.)

104. Through one angular point A of a square ABCD a st. line is drawn meeting the sides BC and DC produced at E and F respectively. If ED and FB intersect in G, show that CG is $\perp$ EF. (Problems—1913.)

105. Prove that the lines

$$ax + (b + c)\,y = b^2 + bc + c^2,$$
$$bx + (c + a)\,y = c^2 + ca + a^2,$$
$$cx + (a + b)\,y = a^2 + ab + b^2,$$

are concurrent, and find the coordinates of their common point. (Problems—1913.)

106. Two circles whose centres are C, C′, touch each other, internally at O. A st. line OPP′ is drawn cutting the circles at P and P′. Show that the locus of the intersection of CP′ and C′P is a circle whose diameter is a harmonic mean between the radii of the given circles; and whose centre is at C″ on the line OCC′ such that OC″ is the harmonic mean between OC and OC′. (Problems—1913.)

107. Prove that the chords of intersection with a fixed circle of all circles through two fixed points are concurrent. (Problems—1912.)

108. Find the equation of two st. lines through the origin and such that the $\perp$s to them from the point (h, k) are $+d$ and $-d$.

109. If axes of reference are drawn on a sheet of paper and if this is folded about the line joining $(1, 3)$ to $(2, 0)$, find the coordinates of the point which falls on (x, y).

Find also the equation of the circle which coincides with $x^2 + y^2 - 2y = 4$. (Problems—1917.)

110. **AS** and **AT**, **BP** and **BQ** are tangents from any two points **A** and **B** to a fixed circle. **C**, **D**, **E**, **F** are the middle points of **AS**, **AT**, **BP**, **BQ** respectively. Prove that **CD** and **EF**, produced if necessary, meet on the line that bisects **AB** at rt. $\angle$s. (Problems—1907.)

ANSWERS

§ 6. (Page 5.)

4. $(0, 0)$, $(2a, 0)$, $(a, a\sqrt{3})$.

5. $(0, 0)$, $(b, 0)$, (b, b), $(0, b)$.

§ 16. (Page 11.)

4. 13·04.

5. 10, 17, 9.

6. (a) $\sqrt{58}$, $\sqrt{82}$, 10; (b) $\frac{1}{2}\sqrt{234}$, $3\sqrt{5}$, $\frac{1}{2}\sqrt{306}$.

7. $\sqrt{82}$, $2\sqrt{10}$, $\sqrt{85}$, $\sqrt{29}$; $7\sqrt{2}$, $\sqrt{137}$.

8. $(0, 0)$ – the origin.

9. $(\frac{8}{3}, \frac{7}{3})$ and $(\frac{13}{3}, \frac{5}{3})$.

11. $\sqrt{a^2+b^2}$.

12. $\frac{4}{9}\sqrt{65}$.

13. 3 or -13.

14. $4x-10y+29=0$.

15. $(\frac{103}{62}, \frac{221}{62})$; 3·7 nearly.

16. $(7, 2)$, $(-\frac{245}{37}, \frac{158}{37})$.

20. $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$.

21. $(\frac{31}{5}, \frac{61}{5})$.

22. $(-18, -14)$.

§ 19. (Page 16.)

3. 36·5.

4. 25·5.

8. 36 miles.

9. 1 : 3.

§ 22. (Page 21.)

2. $y=5x$.

3. (a) The axis of x; (b) The axis of y.

4. $x=4$.

5. $3x-5y=17$.

6. $2x+y+5=0$.

7. $x^2+y^2-8x-6y=0$.

8. $x^2+y^2-2x+4y=31$.

10. $x-2y+3=0$.

§ 24. (Page 23.)

1. (a) $(2, 7)$; (b) $(3, 2)$; (c) $(2, -3)$; (d) $(8, -6)$ and $(-8, 6)$; (e) $(5, 12)$ and $(-\frac{220}{17}, \frac{21}{17})$.

2. $(0, 9)$ and $(-15, 0)$.

3. $(0, 0)$ and $(6, 0)$.

4. $2x=a$.

5. $7x+4y=20$.

6. $x^2+y^2=9$.

§ 27. (Page 27.)

2. (a) $\frac{x}{5}+\frac{y}{2}=1$; (b) $\frac{x}{4}+\frac{y}{6}=-1$; (c) $\frac{x}{3}-\frac{y}{8}=1$.

3. (a) $x-3y=0$; (b) $9x-2y+13=0$; (c) $8x+11y+34=0$.

Intercepts :—(a) 0, 0; (b) $-\frac{13}{9}$, $\frac{13}{2}$; (c) $-\frac{34}{8}$, $-\frac{34}{11}$.

4. $(-5, -3\frac{1}{3})$.

5. $(1\frac{7}{8}, 1\frac{13}{28})$.

6. $(1\frac{1}{3}, 4)$.

7. Ratio of equality.

8. Sides, $x+9y+14=0$; $5x+3y+28=0$, $2x-3y-14=0$;

Medians, $x-5y=14$, $x+2y=-7$, $3x-y=0$;

Centroid $(-1, -3)$.

9. Sides, $2x-5y=-24$, $3x+2y=2$, $x-y=4$, $3x+4y=33$;

Diagonals, $8x-y=18$, $x+9y=34, 87x+438y=5948$;

Line through middle points of diagonals, $2x=5$.

10. $(-6, -7)$, $(5, -2)$, $(-3, 4)$.

§ 40. (Page 38.)

3. (a) $x=\sqrt{3}y$; (b) $y+x\sqrt{3}=0$; (c) $5x-7y+35=0$; (d) $3y\pm 4x+9=0$; (e) $x-y=1$.

4. (a) 9, $-\frac{1}{2}$, $-\frac{12}{5}$); (b) $(\frac{5}{3}, 0)$.

5. $(3, 3\frac{1}{2})$.

6. $x+y+1=0$.

7. $\frac{5}{29}\sqrt{58}$; $\tan^{-1}\frac{7}{3}$.

8. (a) $\mathsf{C}=0$; (b) $\mathsf{A}=0$; (c) $\mathsf{B}=0$; (d) $\mathsf{A}=\mathsf{B}$; (e) $\mathsf{A}+\mathsf{B}=0$.

9. $m=1$.

10. $m=-\frac{1}{9}$, $b=2\frac{7}{9}$.

11. $a=8$, $b=-4$.

13. Intercept $=\frac{\mathsf{C}'-\mathsf{C}}{\mathsf{B}}$.

§ 44. (Page 42.)

1. (a) 45°; (b) 30°; (c) 90°; (d) $\tan^{-1}5$.

3. $9x+4y+47=0$.

5. $2x+3y=14$.

6. $7x+5y=4$.

7. $5x-3y+8=0$.

8. $x-y+2=0$ and $x+y-12=0$.

9. $x-\sqrt{3}y=3\sqrt{3}-5$ and $x+\sqrt{3}y=-3\sqrt{3}-5$.

10. $(2\frac{1}{7}, -4\frac{1}{7})$.

11. $\left(b, \frac{ab-b^2}{c}\right)$.

12. 12 : 5.

14. $(17\sqrt{3}-16)x+47y=344+34\sqrt{3}$; and $(17\sqrt{3}+16)x-47y=34\sqrt{3}-344$.

15. $6x+y+8=0$: and $x-6y=11$.

17. $\left(\frac{a}{2}, \frac{b^2+c^2-ab}{2c}\right)$.

18. $\mathsf{B}x-\mathsf{A}y+p\sqrt{\mathsf{A}^2+\mathsf{B}^2}=0$.

§ 54. (Page 53.)

1. (*a*) $\frac{38}{\sqrt{29}}$; (*b*) $\frac{24}{\sqrt{13}}$; (c) $\frac{12\sqrt{2}}{5}$; ( ) $\frac{5}{\sqrt{41}}$; ( ) $\frac{35}{\sqrt{74}}$.
2. $1\frac{2}{5}$.
3. $\frac{c_2 - c_1}{\sqrt{a^2+b^2}}$.
4. $(-24, 55)$.
5. $8x+6y=15$.
6. $x-y=0$.
7. $2\frac{7}{13}$.
8. $(m_1-m)(ax-by)+b(m_1c-mc_1)+a(c_1-c)=0$.
10. $abc-af^2-bg^2-ch^2+2fgh=0$.
11. $(b-m_1a-c_1)\ (y-mx-c)=(b-ma-c)\ (y-m_1x-c_1)$.
12. $(\mathbf{AC}_1-\mathbf{A}_1\mathbf{C})x+(\mathbf{BC}_1-\mathbf{B}_1\mathbf{C})y=0$.
13. $\sqrt{2}$.
14. $\frac{\pm 2\sqrt{130}-11}{7}$
15. $x+21y=6$ and $189x-9y=692$.
17. $3x+y=2$ and $x-3y=24$.
19. $(\frac{341}{238}, \frac{433}{238})$.
20. $33x+61y=216$.
21. $11\ \frac{(7\sqrt{3}+11)}{13}$.
22. $\frac{6}{5}\sqrt{2}$.
24. $\frac{ab\,(c-d)}{\sqrt{a^2+b^2}}$.
25. $15x+14y=100$.
26. $13x-7y=3$ and $7x+13y=119$.
27. $(24+13\sqrt{3})\,x+23y+52\sqrt{3}+257=0$ and $(24-13\sqrt{3})x+23y-52\sqrt{3}+257=0$.
28. $5x-12y+56=0$ and $5x-12y-74=0$.
29. $6x+y=31$, $x-y+3=0$; $\frac{14}{\sqrt{37}}$, $\frac{7}{\sqrt{2}}$.
30. $(5\frac{10}{13}, 3\frac{9}{13})$.
32. $12x-5y=26$, $y=2$.
33. $\mathbf{A}x+\mathbf{B}y+p\sqrt{\mathbf{A}^2+\mathbf{B}^2}=0$.
34. $\mathbf{B}x-\mathbf{A}y+\mathbf{A}k-\mathbf{B}h=0$.
35. (*a*) $(\frac{306}{121}, -\frac{87}{121})$; (*b*) $(-\frac{175}{121}, -\frac{61}{121})$.

§ 60. (Page 60.)

1. (*a*) $x=a$, $x=b$;
   (*b*) $x-y=0$, $x+y=0$;
   (*c*) $x=0$, $x=3y$;
   (*d*) $2x-y=0$, $4x-3y=0$;
   (*e*) $x=a$, $y=-b$;
   (*f*) $3x-y=-4$, $x-3y=5$.
2. $\frac{2}{3}$ and $\frac{1}{2}$.
3. The axes of coordinates.
4. (*d*) $\tan^{-1}\frac{2}{11}$; (*e*) $90°$; (*f*) $\tan^{-1}\frac{4}{3}$.
5. $bc=ad$.
6. 8.
7. $3x^2-y^2-30x+6y+66=0$.

§ 65. (Page 64.)

1. $2x^2-11xy+12y^2=0$.
2. $x^2+xy=0$.
3. $(-\frac{5}{2}, \frac{9}{2})$; $2(x^2+y^2)=19$.
5. $(\mathsf{A}\sqrt{3}+\mathsf{B})x+(\mathsf{B}\sqrt{3}-\mathsf{A})y+2\mathsf{C}=0$.
6. $2xy+a^2=0$.
8. $9x^2-25y^2=0$.
9. $\tan^{-1}\frac{4}{3}$; $3x^2+2y^2=10$.

§ 66. (Page 66.)

1. (a) $\sqrt{145}$;
   (b) $\sqrt{5a^2+16ab+13b^2}$;
   (c) $a+b$.
2. P $(\frac{5}{8}, 3\frac{3}{8})$; Q $(17\frac{1}{2}, 16\frac{1}{2})$.
3. $\frac{3}{2}ab$.
4. $41\frac{1}{2}$.
5. $10\frac{3}{5}$.
6. $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$.
7. $8x+7y=5$.
9. $2ax+2by=c+d$.
10. $x=y\tan a+b$.
13. $x\cos a+y\sin a=a$.
14. $\tan^{-1}\frac{126}{41}$.
15. $m=\frac{3}{4}$, $b=-4\frac{1}{4}$.
16. $x+y\sqrt{3}+2(\sqrt{3}-1)=0$.
17. $\dfrac{mh-k+a}{\sin a-m\cos a}$.
18. $\dfrac{x}{a}+\dfrac{y}{b}=\dfrac{h}{a}+\dfrac{k}{b}$.
20. $bx+ay=ad$.
21. $(-2, -8)$ and $(-6, -14)$.
23. $a=6$ or -4; $(\frac{1}{2}, \frac{1}{3})$, $(\frac{1}{2}, -\frac{1}{2})$.
25. $45°$.
26. $65x-65y+14a+3b=0$.
27. $b(am+c)x+(bd-ab+adm+ac)y-bd(am+c)=0$.
29. $\dfrac{ab}{\sqrt{a^2+b^2}}$.
30. $x+4y+1=0$.
31. $x+7y+6=0$.
32. $(6\frac{78}{187}, 5\frac{124}{187})$; $(-3\frac{9}{77}, \frac{69}{77})$; $(26\frac{26}{119}, 4\frac{67}{119})$; $(0, 57)$.
33. $(4\frac{7}{10}, -\frac{1}{10})$.
34. $2y-7x\pm x\sqrt{55}=0$.
35. $-\dfrac{19}{\sqrt{153}}$.
37. $\dfrac{\sqrt{21}}{2}$.
38. $x+6y=39$ and $6x-y=12$.
41. $2x+3y+11=0$.
42. $165x+605y+3114=0$; $332x-747y+3466=0$; $497x-142y-2178=0$.
43. $(\frac{2}{3}, \frac{2}{3})$.
44. (a) $(ah-cf)x+(bh-cg)y=0$;
    (b) $(ag-bf)(x-y)+c(f+g)-h(a+b)=0$.
45. $5x-14y=6$ and $154x+55y=229$.
46. $49x-245y=242$.
48. $x+y=10$.
49. $x-y=10$, or $y-x=10$.
50. $7x+5y+50=0$, and $9x+35y+150=0$.
51. $x-4y=11$.
52. $2(a-h)x+2(b-k)y=a^2+b^2-h^2-k^2$.
53. $2x+y+15=0$.
54. $45°$.
55. $10x+4y+11=0$, and $4x-10y-33=0$.
56. $x^2-y^2=0$.
57. -13 or $10\frac{1}{10}$.
59. $(11, 1)$, $(-1, -5)$, or $(-5, 7)$.
61. $-\frac{66}{31}$.
62. 24.

§ 72. (Page 75.)

1. $x^2+y^2=5$.
2. $(x-6)^2+(y-2)^2=9$.
3. $(x+5)^2+(y+1)^2=26$.
4. $a^2+b^2=c^2$.
5. (a) (3, 1), 5; (b) $(-\frac{7}{8}, -\frac{5}{8})$, $\frac{\sqrt{330}}{8}$; (c) (7, 0), 7; (d) $(0, -b)$, $\sqrt{b^2+c^2}$.
8. $x^2+y^2-x-7y=0$.
9. $x^2+y^2-3x=19$.
11. (a) $f=0$; (b) $g=0$.
12. $x^2+y^2-4x+4y=2$.
13. $x^2+y^2-4x+6y=16$.
15. $47(x^2+y^2)-181x+341y-1996=0$.
16. $\sqrt{10}$.
17. $14\sqrt{2}$.
21. $2x-5y=15$.
22. 123.
23. $14(x+y)=17$.
24. $c=c'$.
25. $(h+k)(x^2+y^2)-(h^2+k^2)(x+y)=0$.

§ 81. (Page 85.)

1. (a) $3x+5y=34$; (b) $3x-4y+11=0$; (c) $12x+5y=96$; (d) $gx+fy=0$.
2. $y=x\pm\sqrt{70}$.
3. (a) $\mathsf{A}x+\mathsf{B}y=\pm r\sqrt{\mathsf{A}^2+\mathsf{B}^2}$; (b) $\mathsf{B}x-\mathsf{A}y=\pm r\sqrt{\mathsf{A}^2+\mathsf{B}^2}$.
4. (2, 4).
5. (6, 1).
6. (a) $\mathsf{C}^2=r^2(\mathsf{A}^2+\mathsf{B}^2)$; (b) $(\mathsf{A}g+\mathsf{B}f-\mathsf{C})^2=(\mathsf{A}^2+\mathsf{B}^2)(g^2+f^2-c)$.
7. $a^2b^2=r^2(a^2+b^2)$.
8. $c=g^2$.
9. $x^2+y^2-2(7\pm2\sqrt{5})(x+y)+69\pm28\sqrt{5}=0$.
10. $x-\sqrt{3}y+3\sqrt{3}-1=0$, and $x-\sqrt{3}y-5\sqrt{3}-1=0$.
11. $4\sqrt{\frac{313}{29}}$.
13. $(x-g)\cos a+(y-f)\sin a=r$.
14. $(\frac{13}{5}, -\frac{19}{5})$; $4x+3y+1=0$.
16. $x-3y=3$; $\left(\frac{21\pm3\sqrt{119}}{10}, \frac{-3\pm\sqrt{119}}{10}\right)$.
17. $(\mathsf{A}^2+\mathsf{B}^2)(x^2+y^2)=\mathsf{C}^2$.
18. $34(x^2+y^2)-476x-136y+1753=0$.
19. $(\frac{1}{5}, \frac{7}{5})$; $(17, -13)$.
transverse, $12(5y-7)=(-21\pm5\sqrt{15})(5x-1)$;
direct, $24(y+13)=(-21\pm\sqrt{21})(x-17)$.
20. direct, $y=9$ and $3x+4y+3=0$;
transverse are imaginary.
(2, 9) and $(-1, 0)$ on $x^2+y^2-4x-8y-5=0$;
(5, 9) and $(\frac{7}{5}, -\frac{9}{5})$ on $x^2+y^2-10x-6y-2=0$.
21. $x+3y+g+3f\pm\sqrt{10(g^2+f^2-c)}=0$.
22. $x-\sqrt{3}y\pm10=0$.

§ 92. (Page 94.)

1. (*a*) $3x+5y=30$; (*b*) $ax=r^2$; (*c*) $4x+17=0$; (*d*) $10x-2y=7$; (*e*) $hx+ky=h^2+k^2-r^2$.
2. (*a*) $(2, -7)$; (*b*) $(-\frac{23}{12}, \frac{23}{6})$; (*c*) $(-35, 11)$; (*d*) $(0, 0)$.
3. (*c*) $3x-4y+25=0$; (*e*) $9x+13y=25$; (*f*) $13x-9y=0$; (*g*) $24x-7y=125$.
4. $\left(\frac{c^2}{a}, \frac{c^2}{b}\right)$.
5. $\left(\frac{f^2l-g-lc-fgm}{1+fm+gl}, \frac{g^2m-f-mc-fgl}{1+fm+gl}\right)$.
6. $(1, 4)$.

§ 95. (Page 97.)

1. (*a*) 6; (*b*) 5; (*c*) 8; (*d*) $\sqrt{c}$.
2. $x^2+y^2-10x-4y=7$.
3. $x^2+y^2-16x+51=0$.
4. $4x+3y=25$ and $3x-4y=25$.

§ 98. (Page 98.)

1. $12x+8y=95$.
2. $9x-8y+15=0$.
4. $(-13, -7)$.

Miscellaneous Exercises. (Page 99.)

1. $3x+4y=25$.
2. $2x-11y+329=0$.
3. 113.
4. 29.
5. $\frac{x}{h}+\frac{y}{k}=2$.
6. (*a*) $\frac{x-h}{x_1-x_2}=\frac{y-k}{y_1-y_2}$; (*b*) $(x_1-x_2)(x-h)+(y_1-y_2)(y-k)=0$.
8. 7 : 4.
9. $(\frac{288}{115}, -\frac{141}{115})$.
10. 77.
12. $x^2+y^2=12$.
13. $(a+c, b+d)$.
15. The point $(-a \cos \alpha, -a \sin \alpha)$.
16. $x^2+y^2-6x-3y=0$.
17. $3x-y=3k\pm k\sqrt{10}$.
18. $(\frac{382}{147}, \frac{302}{147})$.
20. $x^2+y^2=hx$.
21. $x^2+y^2-ax-by=0$.
22. $x^2+y^2-gx-fy=0$.
23. $(\frac{12}{13}, \frac{21}{13})$.
24. $8(x^2+y^2)-25x+75y+71=0$.
25. 10 or $\frac{6}{7}$.
26. Centre divides **AB** externally in ratio $k^2 : 1$.
27. (*a*) $(7, 1)$; (*b*) **AD** = **DB**.
30. $(2, 1)$.
31. $\frac{238}{143}$.
32. $x^2+y^2-2h(x-a)-2k(y-b)=a^2+b^2$.

MISCELLANEOUS EXERCISES—*Continued.*

33. $x^2+y^2+2gx+2fy+2c-g^2-f^2=0$.

35. $2x-(b+c)y+2abc=0$.

37. $-\frac{\sqrt{13}}{4}$.

40. $19x^2-60xy+44y^2=0$.

41. $3x^2-8xy-3y^2=0$.

42. $\tan^{-1}\frac{8\sqrt{243}}{23}$.

43. $y=2$ and $15x+8y=31$.

46. $(a+c-g,\ b+d-h)$.

49. $163x+9y+54=0$ and $239x-573y+1112=0$.

67. $x^2+y^2-(hx+ky)=0$.

69. $x^2+y^2-10x+9=0$.

74. $\cos^{-1}\frac{17}{20}$.

76. The circle $3(x^2+y^2)+8x+10y=92$.

77. $3(x^2+y^2)-2a(x+y)=2a^2$.

81. $\frac{c^2-b^2}{2m}$.

85. $y^2-x^2=0$.

86. $x^2+y^2-2a(x+y)+a^2=0$.

88. $x^2+y^2=2(x+y)$.

92. $59(x^2+y^2)-44(x+2y)=740$.

102. $\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=\frac{a^2}{4}$.

105. $-\frac{bc+ca+ab}{a+b+c}$, $\frac{a^2+b^2+c^2+bc+ca+ab}{a+b+c}$.

108. $(k^2-d^2)x^2-2hkxy+(h^2-d^2)y^2=0$.

109. $\frac{18-4x-3y}{5}$, $\frac{6-3x+4y}{5}$, $x^2+y^2-6x-4y+8=0$.